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Finite-Amplitude Inhomogeneous Plane Waves of Exponential Type in Incompressible Elastic Materials

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Abstract. It is proved that elliptically polarized finite-amplitude inhomogeneous plane waves may not propagate in an elastic material subject to the constraint of incompressibility. The waves considered are harmonic in time and exponentially attenuated in a direction distinct from the direction of propagation. The result holds whether the material is stress-free or homogeneously deformed.

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1. Introduction

The problem of wave propagation in a medium is addressed mathematically by seeking solutions for the displacement of a particle to a 'wave equation' characteristic of the medium. Among possible solutions, harmonic forms for the displacement are of great interest because any linear combination of harmonic waves is also a solution of the wave equation. Harmonic waves vary sinusoidally with time and distance, as they travel with constant speed and unchanged profile in a fixed direction. They are called 'homogeneous plane waves' because the displacement field is homogeneous in the planes orthogonal to the direction of propagation.

However, in certain physical contexts, such as gravity waves, surface waves, or reflection and refraction of waves, an attenuation of the amplitude occurs in a direction distinct from the direction of propagation. Thus arises the need to find 'inhomogeneous plane wave' solutions to the wave equation. A simple form for the displacement is that of a vector field $\mathbf{u}(\mathbf{x}, t)$ which varies sinusoidally with frequency ω in the direction of a vector $\mathbf{S}^+$ and is attenuated exponentially in the direction of another vector $\mathbf{S}^-$, so that $\mathbf{u}(\mathbf{x}, t)$ is the real part of the complex quantity $e^{-\omega\mathbf{S}^-\cdot\mathbf{x}}\{\mathbf{A}e^{i\omega(\mathbf{S}^+\cdot\mathbf{x}-t)}\}$, where $\mathbf{A}$ is the amplitude of the wave. The complex vector (or 'bivector' [1]) $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$ is called the 'slowness bivector' and its real and imaginary parts describe the 'planes of constant phase' ($\mathbf{S}^+\cdot\mathbf{x} = \text{constant}$) and the 'planes of constant amplitude' ($\mathbf{S}^-\cdot\mathbf{x} = \text{constant}$). When $\mathbf{S}^+$ and $\mathbf{S}^-$ are parallel, the wave is homogeneous; otherwise, it is inhomogeneous. Similarly, $\mathbf{A}$ is a bivector, whose real and imaginary parts either are parallel (linear polarization) or have distinct directions (elliptical polarization).

In this note, we place ourselves in the context of finite elasticity. Specifically, we are interested in the propagation of plane waves of exponential type in incompressible elastic materials. It has been shown that finite-amplitude homogeneous plane waves (with linear or elliptical polarization) may propagate in deformed incompressible materials (Green [2], Currie and Hayes [3], Boulanger and Hayes [4]). Also, small-amplitude elliptically polarized inhomogeneous plane waves propagating in a deformed incompressible material have received much attention (e.g., Hayes and Rivlin [5], Flavin [6], Belward [7], Belward and Wright [8], Borejko [9], Boulanger and Hayes [10]).

Here we show that *elliptically* polarized *inhomogeneous* plane waves of *finite* amplitude may *not* propagate in any incompressible material, whether deformed or not.

2. Proof

For a deformation bringing a material point from position $\mathbf{X}$ in the reference configuration to position $\mathbf{x}$ in the current configuration, the deformation gradient $\mathbf{F}$ is defined by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}. \tag{1}$$

Because of the incompressibility constraint, any deformation of the material must be isochoric, so that, at all times, we have

$$\det \mathbf{F} = 1. \tag{2}$$

Consider the propagation of an elliptically polarized inhomogeneous plane wave of finite amplitude, which we choose to be of an exponential form. Thus, if $\mathbf{A}$ is the amplitude bivector of the wave and $\mathbf{S} = \mathbf{S}^+ + \mathrm{i}\mathbf{S}^-$ is the slowness bivector with associated frequency ω, then the wave is given by

$$\mathbf{x} = \mathbf{X} + \frac{1}{2}\left\{\mathbf{A}\mathrm{e}^{\mathrm{i}\omega(\mathbf{S}\cdot\mathbf{X}-t)} + \overline{\mathbf{A}}\mathrm{e}^{-\mathrm{i}\omega(\overline{\mathbf{S}}\cdot\mathbf{X}-t)}\right\}, \tag{3}$$

where the modulus of $\mathbf{A}$ is finite and the bar denotes the complex conjugate.

The deformation gradient $\mathbf{F}$ associated with this deformation is given by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{1} + \frac{\omega}{2}\left\{\mathrm{i}\mathbf{A}\otimes\mathbf{S}\mathrm{e}^{\mathrm{i}\omega(\mathbf{S}\cdot\mathbf{X}-t)} - \mathrm{i}\overline{\mathbf{A}}\otimes\overline{\mathbf{S}}\mathrm{e}^{-\mathrm{i}\omega(\overline{\mathbf{S}}\cdot\mathbf{X}-t)}\right\}. \tag{4}$$

Then $J = \det \mathbf{F}$ is given by

$$\begin{aligned} J \;=\; & 1 + \frac{\omega}{2}\left\{\mathrm{i}(\mathbf{A}\cdot\mathbf{S})\mathrm{e}^{\mathrm{i}\omega(\mathbf{S}\cdot\mathbf{X}-t)} - \mathrm{i}\big(\overline{\mathbf{A}}\cdot\overline{\mathbf{S}}\big)\mathrm{e}^{-\mathrm{i}\omega(\overline{\mathbf{S}}\cdot\mathbf{X}-t)}\right\} \\ & - \frac{\omega^2}{4}\left\{\big(\mathbf{A}\cdot\overline{\mathbf{S}}\big)\big(\overline{\mathbf{A}}\cdot\mathbf{S}\big) - (\mathbf{A}\cdot\mathbf{S})\big(\overline{\mathbf{A}}\cdot\overline{\mathbf{S}}\big)\right\}\mathrm{e}^{\mathrm{i}\omega(\mathbf{S}-\overline{\mathbf{S}})\cdot\mathbf{X}}. \end{aligned} \tag{5}$$

However, because $J = 1$ at all times by equation (2), we must have $\mathbf{A} \cdot \mathbf{S} = 0,\ \overline{\mathbf{A}} \cdot \overline{\mathbf{S}} = 0,\ (\overline{\mathbf{A}} \cdot \mathbf{S})(\mathbf{A} \cdot \overline{\mathbf{S}}) - (\mathbf{A} \cdot \mathbf{S})(\overline{\mathbf{A}} \cdot \overline{\mathbf{S}}) = 0$, which imply

$$\begin{cases} \mathbf{A} \cdot \mathbf{S} = \overline{\mathbf{A}} \cdot \mathbf{S} = 0, \\ \mathbf{A} \cdot \overline{\mathbf{S}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{S}} = 0. \end{cases} \tag{6}$$

When the wave is *elliptically* polarized, the amplitude bivector $\mathbf{A}$ is such that $\mathbf{A} \times \overline{\mathbf{A}} \neq \mathbf{0}$ [11]. Then from $(6)_{1,2}$, $\mathbf{S}$ is orthogonal to both $\mathbf{A}$ and $\overline{\mathbf{A}}$, and so parallel to their cross product $\mathbf{A} \times \overline{\mathbf{A}}$. Similarly, from $(6)_{3,4}$, $\overline{\mathbf{S}}$ is also parallel to $\mathbf{A} \times \overline{\mathbf{A}}$, and so $\mathbf{S} \times \overline{\mathbf{S}} = \mathbf{0}$. This is only possible when $\mathbf{S}$ has real direction [1], which means that $\mathbf{S} = k\mathbf{n}$, where k is some complex scalar and $\mathbf{n}$ is a real vector in the common direction of $\mathbf{S}$ and $\overline{\mathbf{S}}$. In this case, the plane wave described by (3) is *homogeneous*.

On the other hand, when the wave is *inhomogeneous*, the slowness bivector $\mathbf{S}$ is such that $\mathbf{S} \times \overline{\mathbf{S}} \neq \mathbf{0}$. Using a similar argument to the above involving $(6)_{1,3}$, we find that $\mathbf{A}$ is parallel to $\mathbf{S} \times \overline{\mathbf{S}}$, and from $(6)_{2,4}$, that also $\overline{\mathbf{A}}$ is parallel to $\mathbf{S} \times \overline{\mathbf{S}}$, leading to $\mathbf{A} \times \overline{\mathbf{A}} = \mathbf{0}$. Then $\mathbf{A} = \alpha\mathbf{a}$, where α is a complex scalar and $\mathbf{a}$ is a real vector in the direction of polarization. In this case, the wave is *linearly* polarized.

Hence, if the wave is *elliptically* polarized ($\mathbf{A} \times \overline{\mathbf{A}} \neq \mathbf{0}$) then it must be *homogeneous*; if the wave is *inhomogeneous* ($\mathbf{S} \times \overline{\mathbf{S}} \neq \mathbf{0}$) then it must be *linearly* polarized.

We conclude that, in an unstrained incompressible material, single trains of *elliptically* polarized *finite*-amplitude *inhomogeneous* plane waves of exponential type may not propagate.

REMARK 1. *Deformed material*. Here, we assume the incompressible material to have been first subjected to a finite homogeneous static triaxial stretch, with stretch ratios $\lambda_1,\ \lambda_2$ and λ_3 (with $\lambda_1\lambda_2\lambda_3 = 1$ to satisfy the incompressibility constraint). Upon this deformation, the inhomogeneous wave was then superposed. Thus, a particle at $\mathbf{X}$ in the reference configuration has moved first to $\widetilde{\mathbf{x}}$ given by $\widetilde{x}_i = \lambda_i X_i\ (i = 1, 2, 3)$, and then to $\mathbf{x}$ given by

$$\mathbf{x} = \widetilde{\mathbf{x}} + \frac{1}{2}\left\{\mathbf{A}\mathrm{e}^{\mathrm{i}\omega(\mathbf{S}\cdot\widetilde{\mathbf{x}}-t)} + \overline{\mathbf{A}}\mathrm{e}^{-\mathrm{i}\omega(\overline{\mathbf{S}}\cdot\widetilde{\mathbf{x}}-t)}\right\}. \tag{7}$$

We see that (7) and (3) are the same, except that $\widetilde{\mathbf{x}}$ replaces $\mathbf{X}$. Therefore, the deformation gradient corresponding to the motion (7) is given by

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \widetilde{\mathbf{x}}}\frac{\partial \widetilde{\mathbf{x}}}{\partial \mathbf{X}} = \widetilde{\mathbf{F}}\mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3), \tag{8}$$

where the tensor $\widetilde{\mathbf{F}}$ is the same as $\mathbf{F}$, with $\widetilde{\mathbf{x}}$ instead of $\mathbf{X}$.

Computation of the determinant $\widetilde{J}$ (say) of the deformation tensor given by (8), yields

$$\widetilde{J} = \det\left(\widetilde{\mathbf{F}}\right)(\lambda_1\lambda_2\lambda_3) = \det\left(\widetilde{\mathbf{F}}\right). \tag{9}$$

Thus, $\widetilde{J}$ is the same as J given by (5), with $\widetilde{\mathbf{x}}$ instead of $\mathbf{X}$. Again, because of the constraint of incompressibility, we must have $\widetilde{J} = 1$ at all times and equations (6) are recovered.

REMARK 2. *Small deformations superposed on large*. Note that in the context of *small*-amplitude waves, terms of second order in the magnitude of the wave's amplitude are negligible when compared to terms of first order. The incompressibility constraint then yields $\mathbf{A} \cdot \mathbf{S} = 0$ (see equation (5)), and does not prevent the wave from being *both* elliptically polarized and inhomogeneous.

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