FINITE-AMPLITUDE INHOMOGENEOUS PLANE WAVES IN A DEFORMED MOONEY–RIVLIN MATERIAL

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Summary

The propagation of finite-amplitude linearly-polarized inhomogeneous transverse plane waves is considered for a Mooney–Rivlin material maintained in a state of finite static homogeneous deformation. It is shown that such waves are possible provided that the directions of the normal to the planes of constant phase and of the normal to the planes of constant amplitude are orthogonal and conjugate with respect to the **B**-ellipsoid, where **B** is the left Cauchy–Green strain tensor corresponding to the initial deformation. For these waves, it is found that even though the system is nonlinear, results on energy flux are nevertheless identical with corresponding results in the classical linearized elasticity theory. Byproducts of the results are new exact static solutions for the Mooney–Rivlin material.

1. Introduction

An isotropic homogeneous elastic material maintained in a state of arbitrary finite static homogeneous deformation exhibits three privileged orthogonal directions, namely those of the principal axes of strain. For a general material held in such a state, Green (**1**) showed that transverse finite-amplitude homogeneous plane waves can travel in a principal direction and Carroll (**2**) showed that homogeneous transverse plane waves of finite amplitude, linearly-polarized in a principal direction, may propagate in a principal plane.

John (**3**) considered the possibility of having finite-amplitude plane waves propagating in *any* direction in a prestressed material; he showed that the so-called Hadamard material was the most general compressible one in which it may happen. Then Currie and Hayes (**4**) extended his result to incompressible bodies and the corresponding material was found to be of the Mooney–Rivlin type, a model used to describe the mechanical behaviour of rubber (**5**). Later, Boulanger and Hayes (**6**, **7**) gave a detailed study of finite-amplitude waves propagating in a deformed Mooney–Rivlin material. For these waves, the displacement is of the form $g(\mathbf{n}\cdot\mathbf{x} - vt)\mathbf{a}$, where g is a function of arbitrary magnitude, $\mathbf{n}$ is a unit vector in the direction of propagation (which may be any direction), v is the speed at which the wave travels and $\mathbf{a}$ is a unit vector in the direction of polarization.

In contrast to this type of wave, *inhomogeneous* plane waves have distinct planes of constant amplitude and of constant phase. The purpose of this paper is to consider the superposition of an inhomogeneous motion upon an arbitrary static homogeneous deformation in a Mooney–Rivlin material. We show that waves with displacement of the form $f(\mathbf{b}\cdot\mathbf{x})g(\mathbf{n}\cdot\mathbf{x} - vt)\mathbf{a}$, where f and g are certain functions of finite magnitude and the vectors $\mathbf{n}$ and $\mathbf{b}$ satisfy a geometric relationship, may propagate in the deformed Mooney–Rivlin material, for *any* orientation of the plane of $\mathbf{b}$ and $\mathbf{n}$.

The plan of the paper is as follows. In section 2, we recall the basic equations describing the Mooney–Rivlin incompressible material. In section 3, we assume that this material is subjected to a finite homogeneous static deformation, upon which a finite-amplitude linearly-polarized inhomogeneous plane wave is superposed.

The equations of motion are derived in section 4 and solved for the functions f and g, the speed v and the pressure. Two types of solutions arise. When the directions of **a**, **b**, and **n** are along principal axes of the initial strain ellipsoid, the solutions are called 'special principal motions' (section 5); when the orientation of the plane containing **b** and **n** is arbitrary, we show that these directions must be conjugate with respect to the elliptical section of the strain ellipsoid corresponding to the initial static deformation by the plane orthogonal to the direction of polarization (section 6).

In section 7, we study a class of solutions which are of interest in some physical contexts (gravity waves, surface waves, waves in layered media, interfacial waves, etc.). For these solutions, the amplitude decays exponentionally in one direction while it varies sinusoidally with time in another. We show how the possible directions of polarization, propagation and attenuation may be constructed geometrically, and give bounds for the values of the phase speed. For these waves, the propagation of energy is also considered. Although the theory is nonlinear, well-known aspects of the linearized theory of wave propagation in conservative media are recovered. In particular, it is shown that the direction of the mean energy flux vector is parallel to the planes of constant amplitude, and that the component of this vector in the direction of propagation is equal to the phase speed times the total energy density.

Finally, in section 8, we investigate the case where the time dependency of the superposed deformation is removed. The corresponding solutions provide examples of inhomogeneous finite static deformations possible in a Mooney–Rivlin material, in the absence of body forces.

2. Basic equations

Here we recall the basic equations governing the deformation of Mooney–Rivlin rubberlike materials.

The constitutive equation for a Mooney–Rivlin material is

$$\mathbf{T} = -p\mathbf{1} + (C + D\mathrm{I})\mathbf{B} - D\mathbf{B}^2 = -(p - D\mathrm{II})\mathbf{1} + C\mathbf{B} - D\mathbf{B}^{-1}. \tag{2.1}$$

Here $\mathbf{T}$ is the Cauchy stress, C and D are constants, $\mathbf{B}$ is the left Cauchy-Green strain tensor defined by $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, where $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient, and the deformation is $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. Also, I and II are invariants of $\mathbf{B}$, given by $\mathrm{I} = \operatorname{tr}\mathbf{B}$ and $\mathrm{II} = \operatorname{tr}\mathbf{B}^{-1}$.

In the constitutive equation (2.1), $p(\mathbf{x}, t)$ corresponds to an arbitrary pressure. Because the material is incompressible, we must have

$$\det\mathbf{F} = 1. \tag{2.2}$$

For this material, the strain energy density Σ, measured per unit volume in the current state of deformation, has the form (**5**)

$$2\Sigma = C(\mathrm{I} - 3) + D(\mathrm{II} - 3). \tag{2.3}$$

It is assumed that

$$C \geqslant 0,\ D > 0, \qquad \text{or} \qquad C > 0,\ D \geqslant 0, \tag{2.4}$$

in order that the strong ellipticity condition be satisfied (see (**7**) for a short proof). If $D = 0$, the material is said to be neo-Hookean which case is being examined elsewhere. Here we exclude this possibility, that is we assume that $(2.4)_1$ holds.

The equations of motion in the absence of body forces are

$$\operatorname{div} \mathbf{T} = \rho \frac{\partial^2 \mathbf{x}}{\partial t^2}, \qquad \frac{\partial T_{ij}}{\partial x_j} = \rho \frac{\partial^2 x_i}{\partial t^2}, \tag{2.5}$$

where ρ is the mass density of the material, measured per unit volume in any configuration (because of the incompressibility constraint).

The energy flux vector $\mathbf{R}$ is defined by

$$\mathbf{R} = -\mathbf{T} \cdot \dot{\mathbf{x}}, \qquad R_i = -T_{ij}\, \dot{x}_j, \tag{2.6}$$

where $\dot{\mathbf{x}}$ is the particle velocity. We write R_α for the rate at which the mechanical energy crosses, at time t, a material element which is normal to the x_α-axis at time t, in the final state of deformation, measured per unit area of surface in this configuration.

Also, the kinetic energy density K measured per unit volume is given by

$$K = \tfrac{1}{2}\rho(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}). \tag{2.7}$$

3. Motion superposed on static homogeneous deformation

Here we consider the propagation of a linearly-polarized inhomogeneous plane wave of finite amplitude in the material, when it is held in a state of finite static homogeneous deformation. We determine the corresponding stresses and energy flux for the motion.

We assume that the material is held in the state of finite static homogeneous deformation given by

$$x_\alpha = \lambda_\alpha X_\alpha, \qquad \alpha = 1, 2, 3, \tag{3.1}$$

in which the particle initially at position $\mathbf{X}$ is displaced to $\mathbf{x}$. The constants $\lambda_1, \lambda_2, \lambda_3$ are such that $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and satisfy $\lambda_1\lambda_2\lambda_3 = 1$, due to (2.2).

In this case, the principal stresses t_α, necessary to support the deformation are given by

$$t_\alpha = -p_0 + (C + D\mathrm{I})\lambda_\alpha^2 - D\lambda_\alpha^4, \qquad \alpha = 1, 2, 3, \tag{3.2}$$

where now $\mathrm{I} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ and p_0 is a constant.

Let a linearly-polarized inhomogeneous plane wave of finite amplitude propagate in the deformed body, so that the final position of the particle, which is initially at $\mathbf{X}$, and at $\mathbf{x}$ in the state of finite static homogeneous deformation, is at $\overline{\mathbf{x}}$ where

$$\overline{\mathbf{x}} = \mathbf{x} + f(\mathbf{b} \cdot \mathbf{x})g(\mathbf{n} \cdot \mathbf{x} - vt)\mathbf{a}. \tag{3.3}$$

Here, ($\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$) form an orthonormal triad. The planes defined by $\mathbf{n} \cdot \mathbf{x} =$ const. are the planes of constant phase and those defined by $\mathbf{b} \cdot \mathbf{x} =$ const. are the planes of constant amplitude. Also, v is the real speed of propagation and (f, g) are two real functions to be determined.

We use bars, for example $\overline{W}$, to denote quantities in the final state of deformation. The deformation gradient $\overline{\mathbf{F}}$ associated with the deformation (3.3) is given by

$$\overline{\mathbf{F}} = \frac{\partial \overline{\mathbf{x}}}{\partial \mathbf{X}} = [\mathbf{1} + f'g\, \mathbf{a} \otimes \mathbf{b} + fg'\, \mathbf{a} \otimes \mathbf{n}]\mathbf{F}, \tag{3.4}$$

where a prime denotes the derivative of a function with respect to its argument.

The corresponding left Cauchy–Green tensor $\overline{\mathbf{B}} = \overline{\mathbf{F}}\,\overline{\mathbf{F}}^{\mathrm{T}}$ and its inverse $\overline{\mathbf{B}}^{-1} = \overline{\mathbf{F}}^{-1\mathrm{T}}\,\overline{\mathbf{F}}^{-1}$ are

$$\begin{aligned}\overline{\mathbf{B}} &= [\mathbf{1} + f'g\,\mathbf{a}\otimes\mathbf{b} + fg'\,\mathbf{a}\otimes\mathbf{n}]\mathbf{B}[\mathbf{1} + f'g\,\mathbf{b}\otimes\mathbf{a} + fg'\,\mathbf{n}\otimes\mathbf{a}],\\ \overline{\mathbf{B}}^{-1} &= [\mathbf{1} - f'g\,\mathbf{b}\otimes\mathbf{a} - fg'\,\mathbf{n}\otimes\mathbf{a}]\mathbf{B}^{-1}[\mathbf{1} - f'g\,\mathbf{a}\otimes\mathbf{b} - fg'\,\mathbf{a}\otimes\mathbf{n}].\end{aligned} \tag{3.5}$$

Also, the invariants of $\overline{\mathbf{B}}$ are given by

$$\begin{aligned}\bar{\mathrm{I}} = \operatorname{tr}\overline{\mathbf{B}} &= \mathrm{I} + 2[f'g(\mathbf{a}\cdot\mathbf{Bb}) + fg'(\mathbf{a}\cdot\mathbf{Bn})]\\ &\quad + (f'g)^2(\mathbf{b}\cdot\mathbf{Bb}) + 2ff'gg'(\mathbf{n}\cdot\mathbf{Bb}) + (fg')^2(\mathbf{n}\cdot\mathbf{Bn}),\\ \overline{\mathrm{II}} = \operatorname{tr}\overline{\mathbf{B}}^{-1} &= \mathrm{II} - 2[f'g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) + fg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n})] + [(f'g)^2 + (fg')^2](\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}).\end{aligned} \tag{3.6}$$

We introduce the coordinates (η, ξ, ζ), given by

$$\eta = \mathbf{n}\cdot\mathbf{x}, \qquad \xi = \mathbf{a}\cdot\mathbf{x}, \qquad \zeta = \mathbf{b}\cdot\mathbf{x}, \tag{3.7}$$

and the coordinates $(\overline{\eta}, \overline{\xi}, \overline{\zeta})$, given by

$$\overline{\eta} = \mathbf{n}\cdot\overline{\mathbf{x}} = \eta, \qquad \overline{\xi} = \mathbf{a}\cdot\overline{\mathbf{x}} = \xi + f(\zeta)g(\eta - vt), \qquad \overline{\zeta} = \mathbf{b}\cdot\overline{\mathbf{x}} = \zeta. \tag{3.8}$$

Now we write the Cauchy stress tensor $\overline{\mathbf{T}}$ in $(\mathbf{n}, \mathbf{a}, \mathbf{b})$. For a Mooney–Rivlin material, the stress–strain relation is given by equation (2.1) or, in the present context, by

$$\overline{\mathbf{T}} = -(\overline{p} - D\overline{\mathrm{II}})\mathbf{1} + C\overline{\mathbf{B}} - D\overline{\mathbf{B}}^{-1}, \tag{3.9}$$

where the arbitrary pressure $\overline{p}$ may be decomposed into the pressure p_0 corresponding to the primary homogeneous deformation and an incremental pressure p^* (say) corresponding to the superposed dynamic deformation: $\overline{p} = p_0 + p^*$. We assume p^* to be of a form similar to the superposed motion (3.3); that is, $p^* = p^*(\mathbf{n}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x}, t) = p^*(\eta, \zeta, t)$.

Using (3.5), (3.6), (3.9), and the notation: $\overline{\mathbf{T}}_{\eta\eta} = \mathbf{n}\cdot\overline{\mathbf{T}}\mathbf{n}$, $\overline{\mathbf{T}}_{\eta\xi} = \mathbf{n}\cdot\overline{\mathbf{T}}\mathbf{a}$, etc., the components of $\overline{\mathbf{T}}$ are found to be

$$\begin{aligned}\overline{\mathbf{T}}_{\eta\eta} &= \mathbf{T}_{\eta\eta} - p^* - 2Df'g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) + D(f'g)^2(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}),\\ \overline{\mathbf{T}}_{\eta\xi} &= \mathbf{T}_{\eta\xi} + C[fg'(\mathbf{n}\cdot\mathbf{Bn}) + f'g(\mathbf{n}\cdot\mathbf{Bb})] + D(fg')^2(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}),\\ \overline{\mathbf{T}}_{\eta\zeta} &= \mathbf{T}_{\eta\zeta} + D[f'g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) + fg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b})] - Dff'gg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}),\\ \overline{\mathbf{T}}_{\xi\xi} &= \mathbf{T}_{\xi\xi} - p^* + 2C[f'g(\mathbf{a}\cdot\mathbf{Bb}) + fg'(\mathbf{a}\cdot\mathbf{Bn})]\\ &\qquad + C[(f'g)^2(\mathbf{b}\cdot\mathbf{Bb}) + 2ff'gg'(\mathbf{n}\cdot\mathbf{Bb}) + (fg')^2(\mathbf{n}\cdot\mathbf{Bn})],\\ \overline{\mathbf{T}}_{\xi\zeta} &= \mathbf{T}_{\xi\zeta} + C[f'g(\mathbf{b}\cdot\mathbf{Bb}) + fg'(\mathbf{n}\cdot\mathbf{Bb})] + Df'g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}),\\ \overline{\mathbf{T}}_{\zeta\zeta} &= \mathbf{T}_{\zeta\zeta} - p^* - 2Dfg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) + D(fg')^2(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}).\end{aligned} \tag{3.10}$$

In passing, we introduce some quantities associated with the energy carried by the disturbance (3.3). First, from (2.7) and (3.3), the kinetic energy density $\overline{K}$ per unit volume is given by

$$\overline{K} = \tfrac{1}{2}\rho v^2(fg')^2. \tag{3.11}$$

Next, the stored-energy density $\overline{W}$ per unit volume associated with the wave is defined, in the absence of body forces, by

$$\overline{W} = \overline{\Sigma} - \Sigma = \tfrac{1}{2}C(\bar{\mathrm{I}} - \mathrm{I}) + \tfrac{1}{2}D(\overline{\mathrm{II}} - \mathrm{II}), \tag{3.12}$$

where $\bar{\mathrm{I}}$ and $\overline{\mathrm{II}}$ are given by (3.6) in our context.

Finally, we introduce the energy flux vector $\overline{\mathbf{R}}$ associated with the motion (3.3). From (2.6) it is given here by $\overline{\mathbf{R}} = -\overline{\mathbf{T}} \cdot \dot{\bar{\mathbf{x}}}$. It is related to the energy flux vector $\mathbf{R}$ measured in the intermediate static state of deformation through (**8**)

$$\mathbf{R} = \left(\frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{x}}\right)^{-1} \overline{\mathbf{R}} = -\left(\frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{x}}\right)^{-1} \overline{\mathbf{T}} \cdot \dot{\bar{\mathbf{x}}}. \tag{3.13}$$

Using (3.3) we have $\partial\bar{\mathbf{x}}/\partial\mathbf{x} = \mathbf{1} + f'g\mathbf{a} \otimes \mathbf{b} + fg'\mathbf{a} \otimes \mathbf{n}$, so that

$$\mathbf{R} = v(fg')[\mathbf{1} - f'g\mathbf{a} \otimes \mathbf{b} - fg'\mathbf{a} \otimes \mathbf{n}]\overline{\mathbf{T}} \cdot \mathbf{a}. \tag{3.14}$$

4. Equations of motion

The equations of motion (2.5) are written as

$$\operatorname{div} \overline{\mathbf{T}} = \rho \frac{\partial^2 \bar{\mathbf{x}}}{\partial t^2}, \qquad \frac{\partial \overline{\mathbf{T}}_{ij}}{\partial \bar{x}_j} = \rho \frac{\partial^2 \bar{x}_i}{\partial t^2}. \tag{4.1}$$

Here div represents the divergence operator with respect to position $\bar{\mathbf{x}}$, that is, with respect to the coordinates $(\bar{\eta}, \bar{\xi}, \bar{\zeta}) = (\eta, \bar{\xi}, \zeta)$. However, by inspection of equations (3.10), we see that $\overline{\mathbf{T}}$ depends on η, ζ and t only, so that div $\overline{\mathbf{T}}$ computed with respect to $\bar{\mathbf{x}}$ is equal to div $\overline{\mathbf{T}}$ computed with respect to $\mathbf{x}$. Hence the equations of motion reduce to

$$\begin{aligned} \overline{\mathbf{T}}_{\eta\eta,\eta} + \overline{\mathbf{T}}_{\eta\zeta,\zeta} &= 0, \\ \overline{\mathbf{T}}_{\xi\eta,\eta} + \overline{\mathbf{T}}_{\xi\zeta,\zeta} &= \rho v^2 fg'', \\ \overline{\mathbf{T}}_{\zeta\eta,\eta} + \overline{\mathbf{T}}_{\zeta\zeta,\zeta} &= 0, \end{aligned} \tag{4.2}$$

or

$$\begin{aligned} &-p^*_{,\eta} + D[f''g(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{n}) - f'g'(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{b})] + D(f'^2 - ff'')gg'(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a}) = 0, \\ &\quad C[f''g(\mathbf{b} \cdot \mathbf{B}\mathbf{b}) + 2f'g'(\mathbf{n} \cdot \mathbf{B}\mathbf{b}) + fg''(\mathbf{n} \cdot \mathbf{B}\mathbf{n})] \\ &\qquad\qquad\qquad + D(f''g + fg'')(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a}) = \rho v^2 fg'', \\ &-p^*_{,\zeta} + D[fg''(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{b}) - f'g'(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{n})] + D(g'^2 - gg'')ff'(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a}) = 0, \end{aligned} \tag{4.3}$$

where commas denote differentiation with respect to the coordinates (η, ξ, ζ); thus, for example, $\overline{\mathbf{T}}_{\zeta\xi,\xi} = \partial\overline{\mathbf{T}}_{\zeta\xi}/\partial\xi$.

Now, equation $(4.3)_2$ is equivalent to

$$[C(\mathbf{b} \cdot \mathbf{B}\mathbf{b}) + D(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a})]f''g + 2C(\mathbf{n} \cdot \mathbf{B}\mathbf{b})f'g' + [C(\mathbf{n} \cdot \mathbf{B}\mathbf{n}) + D(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a}) - \rho v^2]fg'' = 0. \tag{4.4}$$

Also, the second derivatives of p^* must be compatible, that is $p^*_{,\eta\zeta} = p^*_{,\zeta\eta}$, or, from $(4.3)_{1,3}$,

$$f'''g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) - f''g'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) + (f'f'' - ff''')gg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})$$
$$= fg'''(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) - f'g''(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) + (g'g'' - gg''')ff'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}). \quad (4.5)$$

Equations (4.4) and (4.5) are the two equations to be solved for f and g in order that the inhomogeneous motion (3.3) may propagate in the deformed Mooney–Rivlin material. The solutions are established in Appendix A, and a distinction needs be made, according as to whether or not the orthogonal vectors **n**, **a** and **b** are along the principal axes of the **B**-ellipsoid.

5. Special principal motions

Here we present solutions valid when **n**, **a**, and **b** are in the principal directions of the initial strain ellipsoid.

For small-amplitude homogeneous plane waves propagating in a homogeneously deformed body, the term 'principal wave' (**9**) is used to describe a wave travelling along a principal axis of the strain ellipsoid corresponding to the initial static deformation. This terminology can readily be extended to the case of inhomogeneous motions for which the planes of constant phase are orthogonal to such an axis.

We introduce the term 'special principal motion' to describe an inhomogeneous motion for which the normal to the planes of constant phase, the normal to the planes of constant amplitude, and the linear polarization are in the directions of the principal axes of strain. With our notation, a special principal motion is of the form $f(\mathbf{b}\cdot\mathbf{x})g(\mathbf{n}\cdot\mathbf{x} - vt)\mathbf{a}$ where **n**, **a**, and **b** are the vectors **i**, **j**, and **k**.

The most general functions f and g, solutions to the equations of motion (4.4) and (4.5), where, without loss of generality, $\mathbf{n} = \mathbf{i}$, $\mathbf{a} = \mathbf{j}$, and $\mathbf{b} = \mathbf{k}$, are such that one is of exponential type while the other is of sinusoidal type (see proof in Appendix A). For these motions, the quantity v may be arbitrarily prescribed (within an interval).

Explicitly, the possible inhomogeneous special principal motions of the Mooney–Rivlin material are written as

$$\overline{x} = \lambda_1 X, \qquad \overline{y} = \lambda_2 Y + f(\lambda_3 Z)g(\lambda_1 X - vt), \qquad \overline{z} = \lambda_3 Z, \quad (5.1)$$

where either

$$\left.\begin{aligned} f(\lambda_3 Z) &= f(z) = a_1 e^{k\sigma z} + a_2 e^{-k\sigma z}, \\ g(\lambda_1 X - vt) &= g(x - vt) = d_1 \cos k(x - vt) + d_2 \sin k(x - vt), \end{aligned}\right\} \quad (5.2)$$

or

$$\left.\begin{aligned} f(\lambda_3 Z) &= f(z) = a_1 \cos k\sigma z + a_2 \sin k\sigma z, \\ g(\lambda_1 X - vt) &= g(x - vt) = d_1 e^{k(x-vt)} + d_2 e^{-k(x-vt)}. \end{aligned}\right\} \quad (5.3)$$

Here, a_1, a_2, d_1, d_2 are constants, σ is defined by

$$\sigma = \sqrt{(C\lambda_1^2 + D\lambda_2^{-2} - v^2)/(C\lambda_3^2 + D\lambda_2^{-2})}, \quad (5.4)$$

and k and v are arbitrary ($0 \leqslant v^2 < C\lambda_1^2 + D\lambda_2^{-2}$).

6. General solutions

Here, we seek the most general solutions f and g of equation (4.4) satisfying (4.5) such that $f(\mathbf{b}\cdot\mathbf{x})g(\mathbf{n}\cdot\mathbf{x} - vt)\mathbf{a}$ is an *inhomogeneous* motion in the deformed Mooney–Rivlin material, $(\mathbf{a}, \mathbf{n}, \mathbf{b})$ form an orthonormal triad, and the plane of $\mathbf{n}$ and $\mathbf{b}$ is *arbitrary*. It is shown in Appendix A that f and g are the functions defined by either

$$\left.\begin{aligned} f(\zeta) &= a_1 e^{k\zeta} + a_2 e^{-k\zeta}, \\ g(\eta - vt) &= d_1 \cos k(\eta - vt) + d_2 \sin k(\eta - vt), \end{aligned}\right\} \tag{6.1}$$

or

$$\left.\begin{aligned} f(\zeta) &= a_1 \cos k\zeta + a_2 \sin k\zeta, \\ g(\eta - vt) &= d_1 e^{k(\eta - vt)} + d_2 e^{-k(\eta - vt)}. \end{aligned}\right\} \tag{6.2}$$

Here a_1, a_2, d_1, d_2 and k are arbitrary constants, v is such that

$$\rho v^2 = C[(\mathbf{n}\cdot\mathbf{Bn}) - (\mathbf{b}\cdot\mathbf{Bb})], \tag{6.3}$$

and the condition

$$\mathbf{b}\cdot\mathbf{Bn} = 0, \tag{6.4}$$

must be satisfied.

The condition (6.4) means that the unit vectors $\mathbf{n}$ and $\mathbf{b}$ are conjugate with respect to the $\mathbf{B}$-ellipsoid, defined by $\mathbf{x}\cdot\mathbf{Bx} = 1$. Because they are orthogonal, they must lie along the principal axes of the elliptical section of the $\mathbf{B}$-ellipsoid by the plane orthogonal to $\mathbf{a}$. We note that the speed v given by (6.3) is real when $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are in the directions of the minor and major axes of the elliptical section, respectively.

Also note that

$$\rho v^2 = \mathbf{n}\cdot\mathbf{Tn} - \mathbf{b}\cdot\mathbf{Tb} - \frac{\mathbf{b}\cdot\mathbf{Tn}}{\mathbf{b}\cdot\mathbf{B}^{-1}\mathbf{n}}(\mathbf{n}\cdot\mathbf{B}^{-1}\mathbf{n} - \mathbf{b}\cdot\mathbf{B}^{-1}\mathbf{b}), \tag{6.5}$$

so that the speed may be written in terms of the basic strain $\mathbf{B}$ and the corresponding basic stress $\mathbf{T}$.

Finally, using the compatibility equations, the incremental pressure p^* is determined. To within an arbitrary constant, it is found to be either

$$p^* = -D[fg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) + f'g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b})] + \tfrac{1}{2}Dk^2[(d_1^2 + d_2^2)f^2 - 4a_1a_2g^2](\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}), \tag{6.6}$$

when f and g are given by (6.1), or

$$p^* = -D[fg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) + f'g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b})] + \tfrac{1}{2}Dk^2[4d_1^2d_2^2f^2 - (a_1^2 + a_2^2)g^2](\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}), \tag{6.7}$$

when f and g are given by (6.2). Thus, a *finite* displacement of the form $f(\mathbf{b}\cdot\mathbf{x})g(\mathbf{n}\cdot\mathbf{x} - vt)\mathbf{a}$, where $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ is an orthonormal basis, f and g are given by (6.1) or (6.2), v is given by (6.3), and $\mathbf{b}, \mathbf{n}$ satisfy (6.4), is an exact solution to the equations of motion in a homogeneously deformed Mooney–Rivlin material, for any orientation of the plane of $\mathbf{b}$ and $\mathbf{n}$.

7. Sinusoidal evanescent waves

In this section, we restrict our attention to the propagation of a linearly-polarized finite-amplitude inhomogeneous plane wave in a homogeneously deformed Mooney–Rivlin material. The phase of the wave fluctuates sinusoidally in the direction of a unit vector **n**, its amplitude decreases exponentionally in the direction of **b**, orthogonal to **n**, and its polarization is in the direction of **a**, orthogonal to both **n** and **b**. These waves are a subclass of the general solutions to the equations of motion found in the previous section, and arise in a variety of contexts such as Rayleigh waves, Love waves, Stoneley waves, etc.

We give a geometrical construction for the triads (**n**, **a**, **b**) such that the wave may propagate, and present expressions for the phase speed and the pressure. Then we examine the propagation of the energy carried by the wave.

7.1 *Construction*

Henceforth, we consider the motion

$$\bar{\mathbf{x}} = \mathbf{x} + \alpha e^{-k\mathbf{b}\cdot\mathbf{x}}\mathbf{a}\cos k(\mathbf{n}\cdot\mathbf{x} - vt), \qquad (7.1)$$

where α and k are real arbitrary constants, and v is assumed to be real. Recall that in (7.1), $\mathbf{x}$ corresponds to the static homogeneous predeformation $\mathbf{x} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{X}$. As proved in section 6, this motion is possible as long as **n** and **b** are along the minor and major axes of the elliptical section of the **B**-ellipsoid by the plane orthogonal to **a**, respectively.

Boulanger and Hayes (**6**) considered the propagation of finite-amplitude *homogeneous* plane waves in a deformed Mooney–Rivlin material and found that it was possible in any direction of propagation along **n**. Here, we deal with finite amplitude *inhomogeneous* plane waves and it is the *plane* of **n** and **b** (propagation and attenuation directions) that may be arbitrarily prescribed.

Thus we construct a sinusoidal evanescent inhomogeneous plane wave as follows. First, consider any plane passing through the origin. Let **n** and **b** be unit vectors along the respective major and minor axes of the elliptical section of the **B**-ellipsoid by the chosen plane. Then the motion (7.1) is possible in the deformed Mooney–Rivlin material, with a linear polarization in the direction of $\mathbf{a} = \mathbf{b} \wedge \mathbf{n}$, an amplitude exponentially attenuated by factor k in the direction of **b**, and a velocity $v\mathbf{n}$, where $\rho v^2 = C[(\mathbf{n}\cdot\mathbf{Bn}) - (\mathbf{b}\cdot\mathbf{Bb})]$.

In order to write explicit expressions for the directions of **n** and **b**, and for the value of ρv^2, we use a method developed by Boulanger and Hayes (**10**, section 5.7). First we prescribe a plane cutting the **B**-ellipsoid in a central elliptical section. We denote by **a** the unit vector normal to this plane. Then we write the Hamiltonian cyclic decomposition of the tensor **B** as (**10**, section 3.4)

$$\begin{aligned}\mathbf{B} &= \lambda_1^2\,\mathbf{i}\otimes\mathbf{i} + \lambda_2^2\,\mathbf{j}\otimes\mathbf{j} + \lambda_3^2\,\mathbf{k}\otimes\mathbf{k} \\ &= \lambda_2^2\,\mathbf{1} + \tfrac{1}{2}(\lambda_1^2 - \lambda_3^2)(\mathbf{m}^+\otimes\mathbf{m}^- + \mathbf{m}^-\otimes\mathbf{m}^+),\end{aligned} \qquad (7.2)$$

where the unit vectors $\mathbf{m}^{\pm}$ are in the directions of the 'optic axes' of the **B**-ellipsoid and are defined by

$$\sqrt{\lambda_1^2 - \lambda_3^2}\,\mathbf{m}^{\pm} = \sqrt{\lambda_1^2 - \lambda_2^2}\,\mathbf{i} \pm \sqrt{\lambda_2^2 - \lambda_3^2}\,\mathbf{k}. \qquad (7.3)$$

We now seek **n** and **b**, unit vectors in the directions of the principal axes of the elliptical section of the **B**-ellipsoid by the plane $\mathbf{a}\cdot\mathbf{x} = 0$. Hence, **n** and **b** are eigenvectors of the tensor $\Pi\mathbf{B}\Pi$ where

Π, the orthogonal projection upon the plane $\mathbf{a}\cdot\mathbf{x} = 0$, is defined by $\Pi = \mathbf{1} - \mathbf{a}\otimes\mathbf{a}$. With the aid of (7.2), and because $\Pi^2 = \Pi$, we have

$$\Pi\mathbf{B}\Pi = \lambda_2^2\,\Pi + \tfrac{1}{2}(\lambda_1^2 - \lambda_3^2)(\Pi\mathbf{m}^+ \otimes \Pi\mathbf{m}^- + \Pi\mathbf{m}^- \otimes \Pi\mathbf{m}^+). \tag{7.4}$$

Calling $\psi^\pm$ the angles between the polarization direction $\mathbf{a}$ and the optic axes $\mathbf{m}^\pm$ $(0 \leqslant \psi^\pm \leqslant \pi)$, and noting that the vectors $\Pi\mathbf{m}^\pm/\sin\psi^\pm$ are of unit length, we find from (7.4) that $\Pi\mathbf{m}^+/\sin\psi^+ \pm \Pi\mathbf{m}^-/\sin\psi^-$ are eigenvectors of $\Pi\mathbf{B}\Pi$ with eigenvalues $\gamma^\pm$ given by (**10**, section 5.7)

$$\gamma^\pm = \lambda_2^2 + \tfrac{1}{2}(\lambda_1^2 - \lambda_3^2)(\Pi\mathbf{m}^+\cdot\Pi\mathbf{m}^- \pm \sin\psi^+\sin\psi^-). \tag{7.5}$$

Recall that the phase speed v, which is given by equation (6.3),

$$\rho v^2 = C(\mathbf{n}\cdot\mathbf{Bn} - \mathbf{b}\cdot\mathbf{Bb}) = C(\mathbf{n}\cdot\Pi\mathbf{B}\Pi\mathbf{n} - \mathbf{b}\cdot\Pi\mathbf{B}\Pi\mathbf{b}), \tag{7.6}$$

was assumed to be real. Therefore, $\mathbf{n}$ is the eigenvector of $\Pi\mathbf{B}\Pi$ with the greater eigenvalue, which is γ^+ according to (7.5), and $\mathbf{b}$ is the eigenvector of $\Pi\mathbf{B}\Pi$ with the lesser eigenvalue γ^-. Hence, we obtain the propagation and attenuation directions as

$$\begin{aligned} \mathbf{n} &= \Pi\mathbf{m}^+/\sin\psi^+ + \Pi\mathbf{m}^-/\sin\psi^-, \qquad & \mathbf{n}\cdot\mathbf{Bn} &= \gamma^+, \\ \mathbf{b} &= \Pi\mathbf{m}^+/\sin\psi^+ - \Pi\mathbf{m}^-/\sin\psi^-, \qquad & \mathbf{b}\cdot\mathbf{Bb} &= \gamma^-. \end{aligned} \tag{7.7}$$

7.2 *Phase speed and pressure*

We now write the phase speed in terms of the initial stretches and the angles $\psi^\pm$. Upon using (7.5) and (7.7), we have

$$\rho v^2 = C(\lambda_1^2 - \lambda_3^2)\sin\psi^+\sin\psi^-. \tag{7.8}$$

In this connection, it may be noted that for finite-amplitude *homogeneous* plane waves propagating in a homogeneously deformed Mooney–Rivlin material, two linearly-polarized waves exist for each direction of propagation, and the difference between the two corresponding squared speeds is proportional to $(\lambda_1^2 - \lambda_3^2)\sin\phi^+\sin\phi^-$, where $\phi^\pm$ are the angles between the propagation direction and the optic axes of the $\mathbf{B}^{-1}$-ellipsoid (also called the 'acoustic axes') (**6**).

From (7.8), it follows that the minimum value of v is $v_{\min}$ given by

$$v_{\min}^2 = 0, \tag{7.9}$$

and is attained only when $\mathbf{a} = \mathbf{m}^\pm$. In this case, $\mathbf{n}\cdot\mathbf{Bn} = \mathbf{b}\cdot\mathbf{Bb} = \lambda_2^2$, and the plane orthogonal to $\mathbf{a}$ is a plane of central circular section of the $\mathbf{B}$-ellipsoid. In other words, when the plane of $\mathbf{n}$ and $\mathbf{b}$ is prescribed to be orthogonal to an optic axis of the $\mathbf{B}$-ellipsoid, the superposed deformation must be static. We treat this case in section 8.

The maximum value for the phase speed is $v_{\max}$ given by

$$\rho v_{\max}^2 = C(\lambda_1^2 - \lambda_3^2), \tag{7.10}$$

which is attained when $\mathbf{a} = \mathbf{j}$. In that case, $\mathbf{n} = \mathbf{i}$ and $\mathbf{b} = \mathbf{k}$. In other words, the fastest wave occurs for propagation in the direction of the greatest initial stretch. A similar result was established by Ericksen (**11**) for acceleration waves in a homogeneously deformed neo-Hookean material and also by Boulanger and Hayes (**7**) for finite-amplitude homogeneous waves in a homogeneously deformed Mooney–Rivlin material.

We may also write the phase speed in terms of the polarization direction **a** alone. Because $\mathbf{n}\cdot\mathbf{Bb} = 0$ and $\det \mathbf{B} = 1$, we have

$$\begin{aligned}\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a} &= (\mathbf{n}\wedge\mathbf{b})\cdot\mathbf{B}^{-1}(\mathbf{n}\wedge\mathbf{b}) = (\mathbf{n}\wedge\mathbf{b})\cdot(\mathbf{Bn}\wedge\mathbf{Bb})\\ &= (\mathbf{n}\cdot\mathbf{Bn})(\mathbf{b}\cdot\mathbf{Bb}).\end{aligned} \tag{7.11}$$

Writing the trace of **B** in the (**a**, **n**, **b**) basis yields

$$\operatorname{tr}\mathbf{B} = \mathbf{a}\cdot\mathbf{Ba} + \mathbf{n}\cdot\mathbf{Bn} + \mathbf{b}\cdot\mathbf{Bb}. \tag{7.12}$$

Combining (7.11) and (7.12), we see that $\mathbf{n}\cdot\mathbf{Bn}$ and $\mathbf{b}\cdot\mathbf{Bb}$ are the two roots of the following quadratic in r (say),

$$r^2 + [\operatorname{tr}\mathbf{B} - \mathbf{a}\cdot\mathbf{Ba}]r + \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a} = 0, \tag{7.13}$$

so that, using (6.3), v is alternatively given by

$$\rho v^2 = C\sqrt{[\operatorname{tr}\mathbf{B} - (\mathbf{a}\cdot\mathbf{Ba})]^2 - 4(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})}. \tag{7.14}$$

Finally, the pressure p^* is given by

$$\begin{aligned}p^* &= \alpha k D e^{-\mathbf{b}\cdot\mathbf{x}}[(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n})\sin k(\mathbf{n}\cdot\mathbf{x} - vt) + (\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b})\cos k(\mathbf{n}\cdot\mathbf{x} - vt)]\\ &\quad + \tfrac{1}{2}\alpha^2 k^2 D e^{-2k\mathbf{b}\cdot\mathbf{x}}(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}).\end{aligned} \tag{7.15}$$

7.3 *Energy propagation*

Here we consider the energy carried by the wave (7.1). First we compute the total energy density, which is the sum of the kinetic and stored-energy densities, and then the energy flux vector. Our aim is to find a relationship between these two quantities, or more relevantly (as the frequencies of the sinusoidal vibrations may be very high), between the temporal mean values of these quantities. To this effect, we introduce the following notation to designate temporal mean values: if $D(\mathbf{x}, t)$ is a periodic field quantity with frequency ω, then its mean value is $\check{D}$, defined by

$$\check{D} = \frac{\omega}{2\pi}\int_0^{\omega/(2\pi)} D(\mathbf{x}, t)\,dt. \tag{7.16}$$

We begin with the kinetic energy density per unit volume $\overline{K}$ given by equation (3.11). Using (7.1), its mean value $\check{\overline{K}}$ is found to be

$$\check{\overline{K}} = \tfrac{1}{4}\alpha^2 k^2 e^{-2k\mathbf{b}\cdot\mathbf{x}}\rho v^2 = \tfrac{1}{4}\alpha^2 k^2 e^{-2k\mathbf{b}\cdot\mathbf{x}} C[(\mathbf{n}\cdot\mathbf{Bn}) - (\mathbf{b}\cdot\mathbf{Bb})]. \tag{7.17}$$

The stored-energy density $\overline{W}$ per unit volume associated with the wave is given by (3.12), where $\overline{\mathrm{I}}$ and $\overline{\mathrm{II}}$ are given, for the motion (7.1), by

$$\begin{aligned}\overline{\mathrm{I}} &= \mathrm{I} - 2\alpha k e^{-k\xi}[\cos k(\mathbf{n}\cdot\mathbf{x} - vt)(\mathbf{a}\cdot\mathbf{Bb}) + \sin k(\mathbf{n}\cdot\mathbf{x} - vt)(\mathbf{a}\cdot\mathbf{Bn})]\\ &\quad + \alpha^2 k^2 e^{-2k\xi}[\cos^2 k(\mathbf{n}\cdot\mathbf{x} - vt)(\mathbf{b}\cdot\mathbf{Bb}) + \sin^2 k(\mathbf{n}\cdot\mathbf{x} - vt)(\mathbf{n}\cdot\mathbf{Bn})],\\ \overline{\mathrm{II}} &= \mathrm{II} + 2\alpha k e^{-k\xi}[\cos k(\mathbf{n}\cdot\mathbf{x} - vt)(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b})\\ &\quad + \sin k(\mathbf{n}\cdot\mathbf{x} - vt)(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n})] + \alpha^2 k^2 e^{-2k\xi}(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}).\end{aligned} \tag{7.18}$$

Note that because the material is incompressible, the stored-energy density $\overline{W}$ is the same whether it is measured in the reference configuration, in the state of static homogeneous deformation, or in the current configuration. The mean value of this quantity is $\check{\overline{W}}$, given by

$$\check{\overline{W}} = \tfrac{1}{4}\alpha^2 k^2 e^{-2k\mathbf{b}\cdot\mathbf{x}}[C(\mathbf{n}\cdot\mathbf{Bn} + \mathbf{b}\cdot\mathbf{Bb}) + 2D(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})]. \tag{7.19}$$

We can now compute the total energy density $\overline{E}$, which is by definition the sum of the kinetic and stored-energy densities: $\overline{E} = \overline{K} + \overline{W}$. Using (7.17) and (7.19), we write directly the mean value $\check{\overline{E}}$ of the total energy density $\overline{E}$ as

$$\check{\overline{E}} = \tfrac{1}{2}\alpha^2 k^2 e^{-2k\mathbf{b}\cdot\mathbf{x}}[C(\mathbf{n}\cdot\mathbf{Bn}) + D(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})]. \tag{7.20}$$

Now we turn our attention to the energy flux vector, defined in equation (3.14), and find that here, $\mathbf{R}$ is given by

$$\begin{aligned}\mathbf{R} = &-\alpha v k e^{-k\mathbf{b}\cdot\mathbf{x}} \sin k(\mathbf{n}\cdot\mathbf{x} - vt)\{\mathbf{Ta} - \tfrac{1}{2}\alpha^2 k^2 D e^{-2k\mathbf{b}\cdot\mathbf{x}}(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})\mathbf{a}\\ &- \alpha k[C\mathbf{Bb} + D(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})\mathbf{n}]e^{-k\mathbf{b}\cdot\mathbf{x}} \cos k(\mathbf{n}\cdot\mathbf{x} - vt)\\ &- \alpha k[C\mathbf{Bn} + D(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})\mathbf{n}]e^{-k\mathbf{b}\cdot\mathbf{x}} \sin k(\mathbf{n}\cdot\mathbf{x} - vt)\}.\end{aligned} \tag{7.21}$$

The mean value of $\mathbf{R}$ is $\check{\mathbf{R}}$, given by

$$\check{\mathbf{R}} = \tfrac{1}{2}\alpha^2 k^2 v e^{-2k\mathbf{b}\cdot\mathbf{x}}[C\mathbf{Bn} + D(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})\mathbf{n}]. \tag{7.22}$$

Hence, using equation (6.4), we see that

$$\check{\mathbf{R}}\cdot\mathbf{b} = 0, \tag{7.23}$$

which means that the direction of the mean energy flux vector is parallel to the planes of constant amplitude. We also have, using (7.20),

$$\check{\mathbf{R}}\cdot(v^{-1}\mathbf{n}) = \check{\overline{E}}, \tag{7.24}$$

which means that the component of the mean energy flux vector in the direction of propagation is equal to the phase speed times the total energy density.

These results may be compared with results established previously. First, Schouten (**12**, section VII.7) introduced the notion of an energy flux vector for the propagation of elastic waves in anisotropic elastic media. For small homogeneous displacements (that is for displacements of the form $\mathbf{a}\cos(kx - vt)$, where the real vector $\mathbf{a}$ is of infinitesimal magnitude), he proved that the temporal mean values of the energy flux vector and energy densities are related through the same equation as (7.24).

Then Synge (**13**) looked for solutions to the equations of motion in an anisotropic medium in the form $\{\mathbf{A}e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}$ where $\mathbf{A} = \mathbf{A}^+ + i\mathbf{A}^-$ and $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$ are complex vectors, ω is the real frequency and c.c. denotes the complex conjugate. He proved that $\check{\mathbf{R}}\cdot\mathbf{S}^- = 0$ and $\check{\mathbf{R}}\cdot\mathbf{S}^+ \geqslant 0$, where $\check{\mathbf{R}}$ is the time-averaged energy flux vector.

Then Hayes (**14**) showed for *any* linear conservative system, using the same notation as above, that $\check{\mathbf{R}}\cdot\mathbf{S}^- = 0$ and $\check{\mathbf{R}}\cdot\mathbf{S}^+ = \check{E}$, where $\check{E}$ is the mean energy density carried by the wave. In that paper, no assumption is made as to whether or not the medium is anisotropic or subject to an internal constraint such as incompressibility or inextensibility.

Later, Chadwick *et al.* (**15**) and Borejko (**16**) described the dynamics of small amplitude plane waves superposed upon a large homogeneous deformation of a constrained material, be they homogeneous (**15**) or inhomogeneous (**16**). In order to remain consistent with results of linear elastodynamics, these authors (following Lighthill (**17**)) separated energy quantities into 'interaction' and 'incremental' parts. The former terms refer to the interconnection between the primary static stretch and the superposed waves, whereas the latter refer to the waves only, and would not disappear in the absence of prestress. This distinction made, they proved that $\check{\mathbf{R}}^{\text{incr}} \cdot \mathbf{S}^- = 0$ and $\check{\mathbf{R}}^{\text{incr}} \cdot \mathbf{S}^+ = \check{E}^{\text{incr}}$, where the superscript 'incr' is short for 'incremental'.

These results are all similar to those given by (7.23) and (7.24) because, in our context, the displacement $\overline{\mathbf{x}} - \mathbf{x}$ may also be written in the form $\{\mathbf{A}e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}$, with $\mathbf{A} = \alpha\mathbf{a}/2$, $\omega = vk$ and $\mathbf{S} = v^{-1}(\mathbf{n} + i\mathbf{b})$. With this notation, (7.23) and (7.24) are rewritten as $\check{\mathbf{R}} \cdot \mathbf{S}^- = 0$ and $\check{\mathbf{R}} \cdot \mathbf{S}^+ = \check{\overline{E}}$ respectively.

However, the above mentioned studies are situated within the framework of *linearized* theory and it is remarkable that results such as (7.23) and (7.24) may be found in the nonlinear case of a *finite*-amplitude wave propagating in a finitely deformed Mooney–Rivlin material. Note that in the same context, Boulanger and Hayes (**7**) found (7.24) for finite-amplitude *homogeneous* plane waves.

8. Superposed static deformation

An interesting feature of inhomogeneous plane motions is that for certain orientations of the planes of constant phase and constant amplitude, the speed v may be equal to zero (**10**, section 6.3). In this situation, no perturbation propagates, but the static deformation $f(\mathbf{b} \cdot \mathbf{x})g(\mathbf{n} \cdot \mathbf{x})\mathbf{a}$ may nevertheless be superposed upon the primary homogeneous one. This possibility provides new exact solutions to the equations of equilibrium in a Mooney–Rivlin material.

Here, we consider in turn the case of special and non-special principal inhomogeneous deformations.

8.1 *Superposed displacements along principal axes of the **B**-ellipsoid*

For special principal motions, we saw in section 5 that the quantity v can be prescribed within the interval $0 \leqslant v^2 < (C\lambda_1^2 + D\lambda_2^{-2})/\rho$. When we choose v to be zero, we obtain from (5.1) a possible static deformation of a Mooney–Rivlin material. It is written as

$$\overline{x} = \lambda_1 X, \qquad \overline{y} = \lambda_2 Y + f(\lambda_3 Z)g(\lambda_1 X), \qquad \overline{z} = \lambda_3 Z, \tag{8.1}$$

where either

$$\left.\begin{aligned} f(\lambda_3 Z) &= a_1 \exp k\sqrt{\frac{C\lambda_1^2 + D\lambda_2^{-2}}{C\lambda_3^2 + D\lambda_2^{-2}}}\lambda_3 Z + a_2 \exp -k\sqrt{\frac{C\lambda_1^2 + D\lambda_2^{-2}}{C\lambda_3^2 + D\lambda_2^{-2}}}\lambda_3 Z, \\ g(\lambda_1 X) &= d_1 \cos k(\lambda_1 X) + d_2 \sin k(\lambda_1 X), \end{aligned}\right\} \tag{8.2}$$

or

$$\left.\begin{aligned} f(\lambda_3 Z) &= a_1 \cos k\sqrt{\frac{C\lambda_1^2 + D\lambda_2^{-2}}{C\lambda_3^2 + D\lambda_2^{-2}}}\lambda_3 Z + a_2 \sin k\sqrt{\frac{C\lambda_1^2 + D\lambda_2^{-2}}{C\lambda_3^2 + D\lambda_2^{-2}}}\lambda_3 Z, \\ g(\lambda_1 X) &= d_1 \exp k(\lambda_1 X) + d_2 \exp -k(\lambda_1 X). \end{aligned}\right\} \tag{8.3}$$

Here, a_1, a_2, d_1, d_2 and k are arbitrary constants.

The components of the Cauchy stress corresponding to these deformations are as follows:

$$\begin{aligned}
\overline{\mathbf{T}}_{11} &= -\overline{p} + C\lambda_1^2 - D[\lambda_1^{-2} + \lambda_2^{-2}(fg')^2],\\
\overline{\mathbf{T}}_{12} &= (C\lambda_1^2 + D\lambda_2^{-2})fg',\\
\overline{\mathbf{T}}_{13} &= -D\lambda_2^{-2}ff'gg',\\
\overline{\mathbf{T}}_{22} &= -\overline{p} + C[\lambda_1^2(fg')^2 + \lambda_2^2 + \lambda_3^2(f'g)^2] - D\lambda_2^{-2},\\
\overline{\mathbf{T}}_{23} &= (C\lambda_3^2 + D\lambda_2^{-2})f'g,\\
\overline{\mathbf{T}}_{33} &= -\overline{p} + C\lambda_3^2 - D[\lambda_3^{-2} + \lambda_2^{-2}(f'g)^2],
\end{aligned} \tag{8.4}$$

where the pressure $\overline{p}$ is given either by

$$\overline{p} = \tfrac{1}{2}Dk^2\lambda_2^{-2}\left[(d_1^2 + d_2^2)f^2 - 4\left(\frac{C\lambda_1^2 + D\lambda_2^{-2}}{C\lambda_3^2 + D\lambda_2^{-2}}\right)a_1a_2g^2\right], \tag{8.5}$$

when f and g are given by (8.2), or by

$$\overline{p} = -\tfrac{1}{2}Dk^2\lambda_2^{-2}\left[4d_1^2d_2^2f^2 - \left(\frac{C\lambda_1^2 + D\lambda_2^{-2}}{C\lambda_3^2 + D\lambda_2^{-2}}\right)(a_1^2 + a_2^2)g^2\right], \tag{8.6}$$

when f and g are given by (8.3).

8.2 *Superposed displacements in the planes of central circular section of the **B**-ellipsoid*

From section 6, we see that for certain choices of the orthogonal unit vectors **n** and **b**, the quantity v is equal to zero. This is the case when the plane of the conjugate vectors **n** and **b** is a plane of central circular section of the **B**-ellipsoid because then, $\mathbf{n}\cdot\mathbf{Bn} = \mathbf{b}\cdot\mathbf{Bb} = \lambda_2^2$ (see (**10**, section 5.7.1) for instance) and then, from equation (6.3), $v = 0$. The vector **a** is one of the unit vectors $\mathbf{m}^{\pm}$, which are in the directions of the optic axes of the **B**-ellipsoid and are defined by (7.3). Also, (**n**, **b**) is any pair of orthogonal unit vectors in a plane of central circular section $\mathbf{a}\cdot\mathbf{x} = \mathbf{m}^{\pm}\cdot\mathbf{x} = 0$.

To provide an illustrative example, we choose $\mathbf{n} = \mathbf{m}^{\pm}\wedge\mathbf{j}$, $\mathbf{b} = \mathbf{j}$, and $f(\mathbf{b}\cdot\mathbf{x})g(\mathbf{n}\cdot\mathbf{x}) = \alpha e^{-k\mathbf{b}\cdot\mathbf{x}}\cos k\mathbf{n}\cdot\mathbf{x}$, where α is a constant. Hence, two possible finite static deformations of a Mooney–Rivlin material are

$$\left.\begin{aligned}
x &= \lambda_1X + \alpha\sqrt{\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2-\lambda_3^2}}e^{-k\lambda_2Y}\cos k\left(\lambda_1\sqrt{\frac{\lambda_2^2-\lambda_3^2}{\lambda_1^2-\lambda_3^2}}X \mp \lambda_3\sqrt{\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2-\lambda_3^2}}Z\right),\\
y &= \lambda_2Y,\\
z &= \lambda_3Z \pm \alpha\sqrt{\frac{\lambda_2^2-\lambda_3^2}{\lambda_1^2-\lambda_3^2}}e^{-k\lambda_2Y}\cos k\left(\lambda_1\sqrt{\frac{\lambda_2^2-\lambda_3^2}{\lambda_1^2-\lambda_3^2}}X \mp \lambda_3\sqrt{\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2-\lambda_3^2}}Z\right).
\end{aligned}\right\} \tag{8.7}$$

The stress tensor $\overline{\mathbf{T}}$ components for these deformations can be written in the basis $(\mathbf{n}, \mathbf{a}, \mathbf{b}) =$

$(\mathbf{m}^{\pm} \wedge \mathbf{j}, \mathbf{m}^{\pm}, \mathbf{j})$ as

$$\begin{aligned}
\mathbf{n} \cdot \overline{\mathbf{T}}\mathbf{n} &= \mathbf{n} \cdot \mathbf{T}\mathbf{n} - p^* + \alpha^2 k^2 D \lambda_2^4 e^{-2k\zeta} \cos^2 k\eta, \\
\mathbf{n} \cdot \overline{\mathbf{T}}\mathbf{a} &= \mathbf{n} \cdot \mathbf{T}\mathbf{a} - \alpha k (C + D\lambda_2^2) \lambda_2^2 e^{-k\zeta} \sin k\eta, \\
\mathbf{n} \cdot \overline{\mathbf{T}}\mathbf{b} &= \mp \alpha k D \lambda_2^2 \sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} e^{-k\zeta} \cos k\eta - \alpha^2 k^2 D \lambda_2^4 e^{-2k\zeta} \sin k\eta \cos k\eta, \\
\mathbf{a} \cdot \overline{\mathbf{T}}\mathbf{a} &= \mathbf{a} \cdot \mathbf{T}\mathbf{a} - p^* \pm 2\alpha k C \sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} e^{-k\zeta} \sin k\eta + \alpha^2 k^2 C \lambda_2^2 e^{-2k\zeta}, \\
\mathbf{a} \cdot \overline{\mathbf{T}}\mathbf{b} &= -\alpha k (C + D\lambda_2^2) \lambda_2^2 e^{-k\zeta} \cos k\eta, \\
\mathbf{b} \cdot \overline{\mathbf{T}}\mathbf{b} &= \mathbf{b} \cdot \mathbf{T}\mathbf{b} - p^* \pm 2\alpha k D \lambda_2^2 \sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} e^{-k\zeta} \sin k\eta + \alpha^2 k^2 D \lambda_2^4 e^{-2k\zeta} \sin^2 k\eta,
\end{aligned} \tag{8.8}$$

where $\mathbf{T}$ is the constant Cauchy stress tensor defined by (2.1) and the pressure p^* is given by

$$p^* = \pm \alpha k D \lambda_2^2 \sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} e^{-k\zeta} \sin k\eta + \tfrac{1}{2} \alpha^2 k^2 D \lambda_2^4 e^{-2k\zeta}. \tag{8.9}$$

9. Concluding remarks

New solutions to the equations of motion and equilibrium in a deformed Mooney–Rivlin material have been obtained. In the case of a finite exponential sinusoidal wave, the propagation of energy was examined and results formerly established within the framework of linearized elasticity were recovered, although the case of a finite motion is nonlinear.

The waves considered here are *linearly-polarized* and it ought to be noted that this is the only possibility of having finite-amplitude *inhomogeneous* plane waves propagating in a material (deformed or not) subjected to the constraint of incompressibility. This is shown elsewhere (**18**).

Also, the planes of constant phase (orthogonal to **n**) were assumed to be at right angles with the planes of constant amplitude (orthogonal to **b**). We show in Appendix B that for finite-amplitude inhomogeneous plane waves of complex exponential type, this condition must be satisfied.

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APPENDIX A

Solutions for a system of two linear differential equations involving two functions of independent variables

Here, our aim is to find the real functions $f(\zeta)$ and $g(\eta - vt)$ satisfying (4.4) and (4.5), viz

$$\left.\begin{aligned} &\rho v_b^2\, g f'' + 2C(\mathbf{n}\cdot\mathbf{Bb})\, g' f' + \rho(v_n^2 - v^2)\, g'' f = 0,\\ &f''' g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) - f'' g'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) + (f'f'' - ff''')gg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})\\ &\quad = fg'''(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) - f'g''(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) + (g'g'' - gg''')ff'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}),\end{aligned}\right\} \tag{A.1}$$

where

$$\rho v_b^2 = C(\mathbf{b}\cdot\mathbf{Bb}) + D(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}), \qquad \rho v_n^2 = C(\mathbf{n}\cdot\mathbf{Bn}) + D(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}), \tag{A.2}$$

and $\mathbf{B}$ is the left Cauchy–Green strain tensor associated with an arbitrary triaxial stretch. Also, f and g are required to be such that $f(\mathbf{b}\cdot\mathbf{x})g(\mathbf{n}\cdot\mathbf{x} - vt)\mathbf{a}$ (where $(\mathbf{a}, \mathbf{n}, \mathbf{b})$ is an orthonormal triad) is an *inhomogeneous* deformation.

A referee very kindly suggested that the use of a technique by Birkhoff (**19**) would lead to a simplier derivation of the results. Accordingly, this approach is adopted here.

Assuming $\mathbf{n}\cdot\mathbf{Bb} \neq 0$, we can, following Birkhoff (**19**, p. 137), express $(A.1)_1$ in the separated form

$$\sum_{j=1}^{3} F_j(\zeta)G_j(\chi) = 0, \qquad \chi = \eta - vt, \tag{A.3}$$

where $F_1 = f''$, $F_2 = f'$, $F_3 = f$, and $G_1 = \rho v_b^2\, g$, $G_2 = 2C(\mathbf{n}\cdot\mathbf{Bb})\, g'$, $G_3 = \rho(v_n^2 - v^2)\, g''$.

The condition (A.3) is equivalent to the requirement that the vectors $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{G} = (G_1, G_2, G_3)$ be confined to orthogonal subspaces. Calling $\dim\mathbf{F}$ and $\dim\mathbf{G}$ the respective dimensions of the subspaces spanned by the $\mathbf{F}$ and $\mathbf{G}$, we may have three possible cases:

case (i). $\dim\mathbf{F} = \dim\mathbf{G} = 1$;
case (ii). $\dim\mathbf{F} = 1$, $\dim\mathbf{G} = 2$;
case (iii). $\dim\mathbf{F} = 2$, $\dim\mathbf{G} = 1$.

We treat these cases in turn.

Case (i). The dimension of the subspace spanned by the $\mathbf{F}$ is 1 when F_1, F_2, F_3, or equivalently f, f', f'' are proportional. This is possible only when $f(\zeta) = Ae^{\delta\zeta}$, where A and δ are real constants. Similarly, $\dim\mathbf{G} = 1$ yields $g(\eta - vt) = Be^{\epsilon\chi}$ for some real scalars B and ϵ.

However, we now have $fg\mathbf{a} = AB\exp[(\epsilon\mathbf{n}+\delta\mathbf{b})\cdot\mathbf{x}-\epsilon vt)]\mathbf{a}$. Hence, case (i) corresponds to a homogeneous motion and must be discarded in our context.

Case (ii). The condition $\dim\mathbf{F} = 1$ yields

$$f(\zeta) = Ae^{\delta\zeta}, \tag{A.4}$$

where A and δ are real constants. Then equation $(A.1)_1$ reduces to

$$\rho(v_n^2 - v^2)\, g'' + 2C(\mathbf{n}\cdot\mathbf{Bb})\delta\, g' + \rho v_b^2\delta^2\, g = 0, \tag{A.5}$$

This is a linear homogeneous second order differential equation for g, with the following characteristic equation in r (say),

$$\rho(v_n^2 - v^2)\, r^2 + 2C(\mathbf{n}\cdot\mathbf{Bb})\delta\, r + \rho v_b^2\delta^2 = 0. \tag{A.6}$$

If (A.6) has two distinct roots r_1, r_2 (say), then g is of the form

$$g(\chi) = Be^{r_1\chi} + Ce^{r_2\chi}, \tag{A.7}$$

where B and C are constants ($(B, C) \neq (0, 0)$).

Upon using (A.4) and (A.7), equation $(A.1)_2$ yields a linear combination for the independent functions $e^{r_1\chi}$, $e^{r_2\chi}$, $e^{2r_1\chi}$, $e^{2r_2\chi}$, and $e^{(r_1+r_2)\chi}$. Nullity for the coefficient of $e^{(r_1+r_2)\chi}$ yields $r_1 = -r_2$. However, in that case we have $g'' - r_1^2 g = 0$, which, together with (A.5), implies that $\dim\mathbf{G} = 1$.

If (A.6) has a double root r (say), then g is of the form

$$g(\chi) = (B + C\chi)e^{r\chi}, \tag{A.8}$$

where B and C are constants ($(B, C) \neq (0, 0)$).

Substituting (A.4) and (A.8) into $(A.1)_2$ yields a linear combination for the independent functions $e^{r\chi}$, $\chi e^{r\chi}$, $e^{2r\chi}$, and $\chi e^{2r\chi}$. Nullity for the coefficient of $e^{2r\chi}$ yields $C^2 r = 0$, which is not possible in our context.

Case (iii). This case is similar to case (ii), where f, g, $v_n^2 - v^2$ and v_b^2 are replaced by g, f, v_b^2 and $v_n^2 - v^2$ respectively.

We conclude that when $\mathbf{n}\cdot\mathbf{Bb} \neq 0$, no f and g can be found such that $fg\,\mathbf{a}$ is an inhomogeneous deformation. We must therefore assume that

$$\mathbf{b}\cdot\mathbf{Bn} = 0, \tag{A.9}$$

which means that the orthogonal unit vectors $\mathbf{n}$ and $\mathbf{b}$ are conjugate with respect to the $\mathbf{B}$-ellipsoid.

Then, equation $(A.1)_1$ reduces to

$$\frac{v_b^2}{v_n^2 - v^2}\frac{f''(\zeta)}{f(\zeta)} = -\frac{g''(\eta - vt)}{g(\eta - vt)} = \text{const.} = \pm k^2 \quad \text{(say)}. \tag{A.10}$$

Clearly, from (A.2) and the strong ellipticity condition $(2.4)_1$, we have $v_b^2 > 0$ and $v_n^2 > 0$. Now, if $v^2 > v_n^2$, then by (A.10), f''/f and g''/g are constants of the same sign. In this case, f and g would be of the same type (either both hyperbolic or both sinusoidal functions), and again this would lead to a homogeneous motion. We assume therefore that $0 \leqslant v^2 < v_n^2$.

Upon using (A.10), the remaining equation to be satisfied, $(A.1)_2$, reduces to

$$(v_n^2 - v_b^2 - v^2)[f'g(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) - fg'(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b})] = 0. \tag{A.11}$$

Hence, there are two cases: (i) $v^2 \neq v_n^2 - v_b^2$ and (ii) $v^2 = v_n^2 - v_b^2$.

Case (i). if $v^2 \neq v_n^2 - v_b^2$, then by (A.11), we have

$$\frac{f'(\zeta)}{f(\zeta)}(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) = \frac{g'(\eta - vt)}{g(\eta - vt)}(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) = \text{const.} = \mu \quad \text{(say)}. \tag{A.12}$$

Assuming $\mu \neq 0$, (A.12) yields $(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) \neq 0$, $(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) \neq 0$ and also $(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n})^2(f''/f) = (\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b})^2(g''/g) = \mu^2$. However, as noted above, f''/f and g''/g must be constants of opposite signs for an inhomogeneous motion.

Hence, $\mu = 0$. Then, by (A.12), $(\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{n}) = (\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b}) = 0$, which, together with (A.9), implies that **n**, **a** and **b** are along principal directions. In this case, the solutions are either exponential in space and sinusoidal in time,

$$\left.\begin{aligned} f(\zeta) &= a_1 \exp k\sqrt{\frac{v_n^2 - v^2}{v_b^2}}\zeta + a_2 \exp -k\sqrt{\frac{v_n^2 - v^2}{v_b^2}}\zeta,\\ g(\eta - vt) &= d_1 \cos k(\eta - vt) + d_2 \sin k(\eta - vt), \end{aligned}\right\} \tag{A.13}$$

or sinusoidal in space and exponential in time,

$$\left.\begin{aligned} f(\zeta) &= a_1 \cos k\sqrt{\frac{v_n^2 - v^2}{v_b^2}}\zeta + a_2 \sin k\sqrt{\frac{v_n^2 - v^2}{v_b^2}}\zeta,\\ g(\eta - vt) &= d_1 \exp k(\eta - vt) + d_2 \exp -k(\eta - vt). \end{aligned}\right\} \tag{A.14}$$

Here, a_1, a_2, d_1, d_2 are constants, and k and v are arbitrary $(0 \leqslant v^2 < v_n^2)$.

These solutions are valid *only* when **n**, **a** and **b** are along the principal axes of the primary static deformation.

Case (ii). if $v^2 = v_n^2 - v_b^2$, then the solutions are either exponential in space and sinusoidal in time,

$$\left.\begin{aligned} f(\zeta) &= a_1 e^{k\zeta} + a_2 e^{-k\zeta},\\ g(\eta - vt) &= d_1 \cos k(\eta - vt) + d_2 \sin k(\eta - vt), \end{aligned}\right\} \tag{A.15}$$

or sinusoidal in space and exponential in time,

$$\left.\begin{aligned} f(\zeta) &= a_1 \cos k\zeta + a_2 \sin k\zeta,\\ g(\eta - vt) &= d_1 e^{k(\eta - vt)} + d_2 e^{-k(\eta - vt)}. \end{aligned}\right\} \tag{A.16}$$

Here, a_1, a_2, d_1, d_2 are constants, k is arbitrary and v is given by

$$\rho v^2 = C[(\mathbf{n}\cdot\mathbf{B}\mathbf{n}) - (\mathbf{b}\cdot\mathbf{B}\mathbf{b})]. \tag{A.17}$$

These solutions are valid for *any* orientation of the plane of **n** and **b**.

APPENDIX B

Finite-amplitude inhomogeneous plane waves of complex exponential type in deformed Mooney–Rivlin materials

Here, we prove that finite-amplitude inhomogeneous plane waves of complex exponential type may propagate in a homogeneously deformed Mooney–Rivlin material only when the planes of constant phase are at right angles with the planes of constant amplitude.

The material is first subjected to a pure homogeneous static deformation, with corresponding deformation gradient $\mathbb{F}$ and left Cauchy–Green strain tensor $\mathbb{B}$ given by

$$\mathbb{F} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad \mathbb{B} = \operatorname{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2), \qquad \text{with} \quad J = \det \mathbb{F} = \lambda_1\lambda_2\lambda_3. \tag{B.1}$$

Then a linearly-polarized inhomogeneous plane wave of finite amplitude is superposed upon the large static deformation. The motion is given by

$$\overline{\mathbf{x}} = \mathbf{x} + \beta\{e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}\mathbf{a} = \mathbf{x} + 2\beta e^{-\omega\mathbf{S}^-\cdot\mathbf{x}}\cos\omega(\mathbf{S}^+\cdot\mathbf{x} - t)\mathbf{a}. \tag{B.2}$$

Here, β is a finite real scalar, $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$ is a complex vector and ω is the real frequency.

The deformation gradient $\overline{\mathbb{F}}$ associated with the motion (B.2) is given by

$$\overline{\mathbb{F}} = \partial\overline{\mathbf{x}}/\partial\mathbf{X} = \check{\mathbb{F}}\mathbb{F}, \qquad \text{where} \quad \check{\mathbb{F}} = \mathbf{1} + \beta\omega\mathbf{a}\otimes\{ie^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\mathbf{S} + \text{c.c.}\}. \tag{B.3}$$

The left Cauchy–Green tensor is $\overline{\mathbb{B}}$ given by $\overline{\mathbb{B}} = \overline{\mathbb{F}}\,\overline{\mathbb{F}}^{\mathrm{T}} = \check{\mathbb{F}}\mathbb{B}\check{\mathbb{F}}^{\mathrm{T}}$.

The incompressibility constraint demands that

$$\det\overline{\mathbb{F}} = [1 + \beta\omega\{i(\mathbf{a}\cdot\mathbf{S})e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}](\lambda_1\lambda_2\lambda_3) = 1, \tag{B.4}$$

at all times. Therefore we must have

$$\mathbf{a}\cdot\mathbf{S} = 0, \qquad \lambda_1\lambda_2\lambda_3 = 1. \tag{B.5}$$

Also, $\check{J} = \det\check{\mathbb{F}}$ and $\overline{J} = \det\overline{\mathbb{F}}$ are given by

$$\check{J} = 1, \qquad \text{and} \quad \overline{J} = \check{J}\,J = J = \lambda_1\lambda_2\lambda_3 = 1. \tag{B.6}$$

In the absence of body forces, the equations of motion, written in the intermediate configuration, read

$$\mathrm{div}_{\mathbf{x}}\,(\check{J}\,\overline{\mathbb{T}}\check{\mathbb{F}}^{-1\mathrm{T}}) = \overline{\rho}\check{J}\,\frac{\partial^2\overline{\mathbf{x}}}{\partial t^2}, \qquad \frac{\partial(\check{J}\,\overline{\mathbb{T}}\check{\mathbb{F}}^{-1\mathrm{T}})_{ij}}{\partial x_j} = \overline{\rho}\check{J}\,\frac{\partial^2\overline{\mathbf{x}}_i}{\partial t^2}, \tag{B.7}$$

where $\overline{\mathbb{T}}$ is the Cauchy stress tensor associated with motion (B.2). For a Mooney–Rivlin material, $\overline{\mathbb{T}}$ is related to the deformation gradient through

$$\overline{\mathbb{T}} = -\overline{p}\mathbf{1} + C\overline{\mathbb{B}} - D\overline{\mathbb{B}}^{-1}, \tag{B.8}$$

where $\overline{p}$ is the pressure.

Upon using (B.8), we have

$$\check{J}\,\overline{\mathbb{T}}\check{\mathbb{F}}^{-1\mathrm{T}} = -\overline{p}\check{J}\,\check{\mathbb{F}}^{-1\mathrm{T}} + C\check{J}\,\check{\mathbb{F}}\mathbb{B} - D\check{J}\,\overline{\mathbb{B}}^{-1}\check{\mathbb{F}}^{-1\mathrm{T}}. \tag{B.9}$$

Hence, with equation (B.6) and $\partial(\check{J}\,\check{\mathbb{F}}^{-1\mathrm{T}}_{ij})/\partial x_j = 0$, the Euler–Jacobi–Piola identity, the equations of motion (B.7) reduce to

$$\check{\mathbb{F}}^{-1\mathrm{T}}\mathrm{grad}_{\mathbf{x}}\overline{p} = C\mathrm{div}_{\mathbf{x}}(\check{\mathbb{F}}\mathbb{B}) - D\mathrm{div}_{\mathbf{x}}(\overline{\mathbb{B}}^{-1}\check{\mathbb{F}}^{-1\mathrm{T}}) - \rho(\partial^2\overline{\mathbf{x}}/\partial t^2). \tag{B.10}$$

Now we compute in turn the three terms of the right-hand side of this equation. For the first term, we have

$$\begin{aligned} C\mathrm{div}_{\mathbf{x}}(\check{\mathbb{F}}\mathbb{B}) &= C\beta\omega\mathrm{div}_{\mathbf{x}}[\mathbf{a}\otimes\{ie^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\mathbb{B}\mathbf{S} + \text{c.c.}\}] \\ &= -C\beta\omega^2\{(\mathbf{S}\cdot\mathbb{B}\mathbf{S})e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}\mathbf{a}. \end{aligned} \tag{B.11}$$

Using (B.5), the second term is found as

$$\begin{aligned} D\mathrm{div}_{\mathbf{x}}(\overline{\mathbb{B}}^{-1}\check{\mathbb{F}}^{-1\mathrm{T}}) &= D\beta\omega^2\{[(\mathbf{S}\cdot\mathbf{S})\mathbb{B}^{-1}\mathbf{a} + (\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{S})\mathbf{S}]e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\} \\ &\quad + D\beta^2\omega^3(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{a})\{i(\mathbf{S}\cdot\mathbf{S})e^{2i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\mathbf{S} + \text{c.c.}\} \\ &\quad + D\beta^2\omega^3(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{a})e^{i\omega(\mathbf{S}-\widetilde{\mathbf{S}})\cdot\mathbf{x}}\{i[(\mathbf{S}-\widetilde{\mathbf{S}})\cdot\widetilde{\mathbf{S}}]\mathbf{S} + \text{c.c.}\}. \end{aligned} \tag{B.12}$$

Finally, the third term is given by

$$-\rho(\partial^2\overline{\mathbf{x}}/\partial t^2) = \rho\beta\omega^2\mathbf{a}\{e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\}. \tag{B.13}$$

In view of (B.11), (B.12), and (B.13), we must take $\overline{p}$ (to within a constant term) of the form

$$\overline{p} = \beta\omega\{ip_1 e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\} + \beta^2\omega^2\{p_2 e^{2i\omega(\mathbf{S}\cdot\mathbf{x}-t)} + \text{c.c.}\} + \beta^2\omega^2 p_3 e^{i\omega(\mathbf{S}-\widetilde{\mathbf{S}})\cdot\mathbf{x}}, \tag{B.14}$$

where p_1, p_2, p_3 are scalars (p_3 is real). With this decomposition, we compute the right-hand side of (B.10) as

$$\begin{aligned}-\check{\mathbb{F}}^{-1\mathrm{T}}\text{grad}_{\mathbf{x}}\overline{p} \quad = \quad & \beta\omega^2\{p_1 e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\mathbf{S} + \text{c.c.}\} - \beta^2\omega^3\{ip_2 e^{2i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\mathbf{S} + \text{c.c.}\} \\ & - \beta^2\omega^3 p_3 i(\mathbf{S}-\widetilde{\mathbf{S}})e^{i\omega(\mathbf{S}-\widetilde{\mathbf{S}})\cdot\mathbf{x}}.\end{aligned} \tag{B.15}$$

Now, using (B.11), (B.12), (B.13) and (B.15), we write the equations of motion (B.10), separating the respective coefficients of $e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}$, $e^{2i\omega(\mathbf{S}\cdot\mathbf{x}-t)}$, and $e^{i\omega(\mathbf{S}-\widetilde{\mathbf{S}})\cdot\mathbf{x}}$, as

$$\begin{aligned}&-p_1\mathbf{S} + C(\mathbf{S}\cdot\mathbb{B}\mathbf{S})\mathbf{a} + D[(\mathbf{S}\cdot\mathbf{S})\mathbb{B}^{-1}\mathbf{a} + (\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{S})\mathbf{S}] = \rho\mathbf{a}, \\ &-p_2\mathbf{S} - D(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{a})(\mathbf{S}\cdot\mathbf{S})\mathbf{S} = 0, \\ &-p_3(\mathbf{S}-\widetilde{\mathbf{S}}) + D(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{a})\{[(\mathbf{S}-\widetilde{\mathbf{S}})\cdot\widetilde{\mathbf{S}})]\mathbf{S} + [(\mathbf{S}-\widetilde{\mathbf{S}})\cdot\mathbf{S})]\widetilde{\mathbf{S}}\} = 0.\end{aligned} \tag{B.16}$$

Writing $\mathbf{S}$ as $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$, equation $(\text{B.16})_3$ is equivalent to

$$[p_3 - 2D(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{a})(\mathbf{S}^-\cdot\mathbf{S}^-)]\mathbf{S}^- = [2D(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{a})(\mathbf{S}^-\cdot\mathbf{S}^+)]\mathbf{S}^+. \tag{B.17}$$

Now, because the motion is inhomogeneous, $\mathbf{S}^-$ is not parallel to $\mathbf{S}^+$ and therefore, we must have $p_3 - 2D(\mathbf{a}\cdot\mathbb{B}^{-1}\mathbf{a})(\mathbf{S}^-\cdot\mathbf{S}^-) = 0$ and also

$$\mathbf{S}^-\cdot\mathbf{S}^+ = 0, \tag{B.18}$$

which means that the planes of constant phase are orthogonal to the planes of constant amplitude.