# The explicit secular equation for surface acoustic waves in monoclinic elastic crystals

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The secular equation for surface acoustic waves propagating on a monoclinic elastic half-space is derived in a direct manner, using the method of first integrals. Although the motion is at first assumed to correspond to generalized plane strain, the analysis shows that only two components of the mechanical displacement and of the tractions on planes parallel to the free surface are nonzero. Using the Stroh formalism, a system of two second order differential equations is found for the remaining tractions. The secular equation is then obtained as a quartic for the squared wave speed. This explicit equation is consistent with that found in the orthorhombic case. The speed of subsonic surface waves is then computed for 12 specific monoclinic crystals. © *2001 Acoustical Society of America.* [DOI: 10.1121/1.1356703]

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## I. INTRODUCTION

The modern theory of surface acoustic waves in anisotropic media owes most of its results to the pioneering works of A. N. Stroh. Although his two seminal articles[1,2] went largely unnoticed for a long time, their theoretical implications were far reaching, as many came to realize since their publication. Among others, Currie,[3] Barnett and Lothe,[4] Chadwick and Smith[5] were able to use his ''sextic formalism'' to address many problems, such as the existence of a single real secular equation for the wave speed, the existence of a limiting velocity (the smallest velocity of body wave solutions) which defines ''subsonic'' and ''supersonic'' ranges for the speeds, or numerical schemes to compute the polarization vectors and the speed of the surface wave. A comprehensive review of these topics can be found in a textbook by Ting.[6]

However precise numerical procedures might be, there is still progress to be made in the search for secular equations in analytic form. So far, explicit expressions have remained few. The secular equation for surface waves in orthorhombic crystals was established by Sveklo[7] as early as 1948 and later, Royer and Dieulesaint[8] proved that it could account for 16 different crystal configurations, such as tetragonal, hexagonal, or cubic. For monoclinic media, Chadwick and Wilson[9] devised a procedure to derive the secular equation, which is given as ''explicit, [...] *apart* from the solution of [a] bicubic equation.'' The object of this paper is to derive *one* expression for the secular equation which is *fully explicit*, when the surface wave propagates in monoclinic crystals.

A classical approach to the problem of surface waves in anisotropic crystals is to consider that a wave propagates with speed $v$ in the direction $x_1$ of a material axis (on the free plane surface) of the material, and is attenuated along another material axis $x_2$, orthogonal to the free surface, so that the mechanical displacement $\mathbf{u}$ is written as $\mathbf{u}=\mathbf{u}(x_1+px_2-vt)$, where $p$ is unknown. Then, assuming a complex exponential form for the displacement, the equations of motion are written in the absence of body forces and solved for $p$. Finally, the boundary conditions yield the secular equation for $v$. The principal mathematical difficulty arising from this procedure is that the equations of motion yield a sextic (generalized plane strain) or a quartic (plane strain) for $p$ which in general are impractical to solve analytically, or even, as a numerical scheme suggests in the sextic case, are actually insoluble analytically (in the sense of Galois).[10]

In 1994, Mozhaev[11] proposed ''some new ideas in the theory of surface acoustic waves.'' He introduced a novel method based on first integrals[12] of the displacement components, which bypasses the sextic (or quartic) equation for $p$ and yields directly the secular equation. He successfully applied this method to the case of orthorhombic materials. In the present paper, generalized plane strain surface waves in a monoclinic crystal with plane of symmetry at $x_3=0$ are examined. The method of first integrals is adapted in order to be applied to the tractions components on the planes $x_3=\text{const}$, rather than to the displacement components. This switch presents several advantages. First, the equations of motion, the boundary conditions, and eventually the secular equation itself, are expressed directly in terms of the usual elastic stiffnesses. Second, it makes it apparent that one of the traction components is zero and thus that, in this paper's context, generalized plane strain leads to plane stress. Third, the boundary conditions are written in a direct and natural manner, because they correspond to the vanishing of the tractions on the free surface and at infinite distance from this surface. Finally, this procedure can easily accommodate an internal constraint, such as incompressibility[13,14] (the secular equation for surface waves in incompressible monoclinic linearly elastic materials is obtained elsewhere).

The plan of the paper is the following. After a brief review of the basic equations describing motion in linearly elastic monoclinic materials (Sec. II), the equations of motion are written down in Sec. III for a surface acoustic wave with three displacement components which depend on two

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coordinates, that in the direction of propagation and that in the direction normal to the free surface (generalized plane strain). Then in Sec. IV, it is seen that one of the traction components is identically zero (plane stress), and that consequently, so is one of the displacement components (plane strain). For the remaining two traction components, coupled equations of motion and the boundary conditions are derived in Sec. V. Finally in Sec. VI, the method of first integrals is applied and the secular equation for acoustic surface waves in monoclinic elastic materials is derived explicitly. As a check, the subcase of orthorhombic materials is treated, and numerical results obtained by Chadwick and Wilson[9] for some monoclinic materials are recovered.

## II. PRELIMINARIES

First, the governing equations for a monoclinic elastic material are recalled. The material axes of the media are denoted by $x_1$, $x_2$, and $x_3$, and the plane $x_3=0$ is assumed to be a plane of material symmetry. For such a material, the relationship between the nominal stress $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\epsilon}$ is given by[15]

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16}\\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26}\\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36}\\ 0 & 0 & 0 & c_{44} & c_{45} & 0\\ 0 & 0 & 0 & c_{45} & c_{55} & 0\\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{31}\\ 2\epsilon_{12}\end{bmatrix}, \tag{1}$$

where $c$'s denote the elastic stiffnesses, and the strain components $\boldsymbol{\epsilon}$'s are related to the displacement components $u_1$, $u_2$, $u_3$ through

$$2\epsilon_{ij}=(u_{i,j}+u_{j,i}) \quad (i,j=1,2,3). \tag{2}$$

The equations of motion, written in the absence of body forces, are

$$\sigma_{ij,j}=\rho u_{i,tt} \quad (i=1,2,3), \tag{3}$$

where $\rho$ is the mass density of the material, and the comma denotes differentiation.

Finally, the $6\times 6$ matrix $\mathbf{c}$ given in Eq. (1) must be positive definite in order for the strain-energy function density to be positive.

## III. SURFACE WAVES

Now the propagation of a surface wave on a semi-infinite body of monoclinic media is modeled. In the same manner as Mozhaev,[11] the amplitude of the associated displacement is assumed to be varying sinusoidally with time in the direction of propagation $x_1$, while its variation in the direction $x_2$, orthogonal to the free surface, is not stated explicitly. Thus calling $v$ the speed of the wave, and $k$ the associated wave number, the displacement components are written in the form

$$u_j(x_1,x_2,x_3,t)=U_j(x_2)e^{ik(x_1-vt)} \quad (j=1,2,3), \tag{4}$$

where the $U$'s depend on $x_2$ only. For these waves, the planes of constant phase are orthogonal to the $x_1$-axis, and the planes of constant amplitude are orthogonal to the $x_2$-axis.

The stress-strain relations (1) reduce to

$$\begin{aligned}&t_{11}=ic_{11}U_1+c_{12}U_2'+c_{16}(U_1'+iU_2),\\ &t_{22}=ic_{12}U_1+c_{22}U_2'+c_{26}(U_1'+iU_2),\\ &t_{33}=ic_{13}U_1+c_{23}U_2'+c_{36}(U_1'+iU_2),\\ &t_{32}=c_{44}U_3'+ic_{45}U_3, \quad t_{13}=c_{45}U_3'+ic_{55}U_3,\\ &t_{12}=ic_{16}U_1+c_{26}U_2'+c_{66}(U_1'+iU_2),\end{aligned} \tag{5}$$

where the prime denotes differentiation with respect to $kx_2$, and the $t$'s are defined by

$$\sigma_{ij}(x_1,x_2,x_3,t)=kt_{ij}(x_2)e^{ik(x_1-vt)} \quad (i,j=1,2,3). \tag{6}$$

The boundary conditions of the problem (surface $x_2=0$ free of tractions, vanishing displacement as $x_2$ tends to infinity) are

$$t_{i2}(0)=0, \quad U_i(\infty)=0 \quad (i=1,2,3). \tag{7}$$

Finally, the equations of motion (3) reduce to

$$\begin{aligned}&it_{11}+t_{12}'=-\rho v^2U_1, \quad it_{12}+t_{22}'=-\rho v^2U_2,\\ &it_{13}+t_{32}'=-\rho v^2U_3.\end{aligned} \tag{8}$$

At this point, a sextic formalism could be developed for the three displacement components $U_1$, $U_2$, $U_3$, and the three traction components $t_{12}$, $t_{22}$, $t_{32}$. However, it turns out that one of these traction components is identically zero, as is now proved.

## IV. PLANE STRESS

It is known (see the Appendix of Stroh's 1962 paper,[2] and also Ting's book,[6] p. 66) that for a two-dimensional deformation of a monoclinic crystal with axis of symmetry at $x_3=0$, the displacements $u_1$ and $u_2$ are decoupled from $u_3$. Taking $u_3=0$ for surface waves, it follows from the stress–strain relationships (5) that $t_{13}=t_{32}=0$. Here, an alternative proof of this result is presented.

Using Eqs. $(5)_4$, $(8)_3$, and $(5)_5$, two first order differential equations for $t_{32}$ and $U_3$ are found as

$$t_{32}=ic_{45}U_3+c_{44}U_3', \quad t_{32}'=(c_{55}-\rho v^2)U_3-ic_{45}U_3'. \tag{9}$$

These equations may be inverted to give $U_3$ and $U_3'$ as

$$\begin{aligned}&(c_{44}c_{55}-c_{45}^2-c_{44}\rho v^2)U_3=ic_{45}t_{32}+c_{44}t_{32}',\\ &(c_{44}c_{55}-c_{45}^2-c_{44}\rho v^2)U_3'=(c_{55}-\rho v^2)t_{32}-ic_{45}t_{32}'.\end{aligned} \tag{10}$$

Differentiation of Eq. $(10)_1$ and comparison with Eq. $(10)_2$ yields the following second order differential equation for $t_{32}$:

$$c_{44}t_{32}''+2ic_{45}t_{32}'-(c_{55}-\rho v^2)t_{32}=0. \tag{11}$$

The boundary conditions (7) and Eq. $(5)_4$ imply that the stress component $t_{32}$ must satisfy $t_{32}(0)=t_{32}(\infty)=0$. The only solution of this boundary value problem for the differential equation (11) is the trivial one. Consequently,

$$t_{32}(x_2)=0 \quad \text{for all} \quad x_2, \tag{12}$$

and so it is proved that, as far as the propagation of surface acoustic waves in monoclinic crystals with plane of symmetry at $x_3=0$ is concerned, generalized plane strain leads to plane stress.

It is also worth noting that by Eq. $(10)_1$, plane stress leads in turn to plane strain which, as an assumption, was not needed a priori. This result was obtained by Stroh[2] in a different manner: ''[when] there is a reflection plane normal to the $x_3$ axis, [...] there is no coupling of the displacement $u_3$ with $u_1$ and $u_2$; any two dimensional problem reduces to one of plane strain $(u_3=0)$ and one of anti-plane strain $(u_1=u_2=0)$.''

Now the equations of motion can be written for the remaining displacements and traction components.

## V. EQUATIONS OF MOTION

Here, the equations of motion are derived, first as a system of four first order differential equations for the nonzero components of mechanical displacement and tractions, and then as a system of two second order differential equations for the tractions.

The stress–strain relations (5) and the equations of motion (8) lead to a system of differential equations for the displacement components $U_1$, $U_2$, and for the traction components $t_1$, $t_2$, defined by

$$t_1=t_{12}, \quad t_2=t_{22}. \tag{13}$$

This system is as follows

$$\begin{bmatrix} \mathbf{u}' \\ \mathbf{t}' \end{bmatrix} = \begin{bmatrix} i\mathbf{N}_1 & \mathbf{N}_2 \\ -(\mathbf{N}_3+X\mathbf{1}) & i\mathbf{N}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \end{bmatrix}, \tag{14}$$

where $\mathbf{u}=[U_1,U_2]^T$, $\mathbf{t}=[t_1,t_2]^T$, $X=\rho v^2$, and the $2\times 2$ matrices $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_3$ are submatrices of the fundamental elasticity matrix $\mathbf{N}$, introduced by Ingebrigsten and Tonning.[16] Explicitly, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ are given by[6]

$$\begin{gathered} -\mathbf{N}_1=\begin{bmatrix} r_6 & 1 \\ r_2 & 0 \end{bmatrix}, \quad \mathbf{N}_2=\begin{bmatrix} s_{22} & -s_{26} \\ -s_{26} & s_{66} \end{bmatrix}=\mathbf{N}_2^T, \\ -\mathbf{N}_3=\begin{bmatrix} \eta & 0 \\ 0 & 0 \end{bmatrix}=-\mathbf{N}_3^T, \end{gathered} \tag{15}$$

where the quantities $r_2$, $r_6$, $s_{22}$, $s_{26}$, $s_{66}$, and $\eta$ are given in terms of the elastic stiffnesses as

$$\begin{gathered} \Delta=\begin{vmatrix} c_{22} & c_{26} \\ c_{26} & c_{66} \end{vmatrix}=c_{22}c_{66}-c_{26}^2, \\ r_6=\frac{1}{\Delta}(c_{22}c_{16}-c_{12}c_{26}), \quad r_2=\frac{1}{\Delta}(c_{12}c_{66}-c_{16}c_{26}), \\ s_{ij}=\frac{1}{\Delta}c_{ij} \quad (i,j=2,6), \\ \eta=\frac{1}{\Delta}\begin{vmatrix} c_{11} & c_{12} & c_{16} \\ c_{12} & c_{22} & c_{26} \\ c_{16} & c_{26} & c_{66} \end{vmatrix} \\ =c_{11}-\frac{c_{66}c_{12}^2+c_{22}c_{16}^2-2c_{12}c_{16}c_{26}}{c_{22}c_{66}-c_{26}^2}. \end{gathered} \tag{16}$$

Throughout the paper, it is assumed that the matrix $\mathbf{N}_3+X\mathbf{1}$ is not singular, which means that the surface wave propagates at a speed distinct from that given by $\rho v^2=\eta$. This assumption made, the second vector line of the system (14) yields

$$\mathbf{u}=i(\mathbf{N}_3+X\mathbf{1})^{-1}\mathbf{N}_1^T\mathbf{t}-(\mathbf{N}_3+X\mathbf{1})^{-1}\mathbf{t}'. \tag{17}$$

On the other hand, differentiation of the system (14) leads to

$$\begin{bmatrix} \mathbf{u}'' \\ \mathbf{t}'' \end{bmatrix} = \begin{bmatrix} -\mathbf{N}_1\mathbf{N}_1-\mathbf{N}_2(\mathbf{N}_3+X\mathbf{1}) & i(\mathbf{N}_1\mathbf{N}_2+\mathbf{N}_2\mathbf{N}_1^T) \\ -i[(\mathbf{N}_3+X\mathbf{1})\mathbf{N}_1+\mathbf{N}_1^T(\mathbf{N}_3+X\mathbf{1})] & -(\mathbf{N}_3+X\mathbf{1})\mathbf{N}_2-\mathbf{N}_1^T\mathbf{N}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \end{bmatrix}. \tag{18}$$

Now the second vector line of this equation yields, using Eq. (17), a system of two second order differential equations for $\mathbf{t}$, written as

$$\hat{\alpha}_{ik}t_k''-i\hat{\beta}_{ik}t_k'-\hat{\gamma}_{ik}t_k=0, \tag{19}$$

where the symmetric $2\times 2$ matrices $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, and $\hat{\boldsymbol{\gamma}}$, are given by

$$\begin{gathered} \hat{\boldsymbol{\alpha}}=-(\mathbf{N}_3+X\mathbf{1})^{-1}, \\ \hat{\boldsymbol{\beta}}=-\mathbf{N}_1(\mathbf{N}_3+X\mathbf{1})^{-1}-(\mathbf{N}_3+X\mathbf{1})^{-1}\mathbf{N}_1^T, \\ \hat{\boldsymbol{\gamma}}=\mathbf{N}_2-\mathbf{N}_1(\mathbf{N}_3+X\mathbf{1})^{-1}\mathbf{N}_1^T, \end{gathered} \tag{20}$$

or, explicitly, by their components,

$$\begin{gathered} \hat{\alpha}_{11}=\frac{1}{\eta-X}, \quad \hat{\alpha}_{12}=0, \quad \hat{\alpha}_{22}=-\frac{1}{X}, \\ \hat{\beta}_{11}=-\frac{2r_6}{\eta-X}, \quad \hat{\beta}_{12}=\frac{1}{X}-\frac{r_2}{\eta-X}, \quad \hat{\beta}_{22}=0, \\ \hat{\gamma}_{11}=s_{22}+\frac{r_6^2}{\eta-X}-\frac{1}{X}, \quad \hat{\gamma}_{12}=\frac{r_2r_6}{\eta-x}-s_{26}, \\ \hat{\gamma}_{22}=\frac{r_2^2}{\eta-X}+s_{66}. \end{gathered} \tag{21}$$

The system (19) of second order differential equations for the traction components is more convenient to work with than the corresponding system for the displacement compo-

TABLE I. Values of the relevant elastic stiffnesses (GPa), density (kg m$^{-3}$), and surface wave speed (m s$^{-1}$) for 12 monoclinic crystals.

| Material | $c_{11}$ | $c_{22}$ | $c_{12}$ | $c_{16}$ | $c_{26}$ | $c_{66}$ | $\rho$ | $v$ |
|---|---|---|---|---|---|---|---|---|
| aegirite-augite | 216 | 156 | 66 | 19 | 25 | 46.5 | 3420 | 3382 |
| augite | 218 | 182 | 72 | 25 | 20 | 51.1 | 3320 | 3615 |
| diallage | 211 | 154 | 37 | 12 | 15 | 62.2 | 3300 | 4000 |
| diopside | 238 | 204 | 88 | −34 | −19 | 58.8 | 3310 | 3799 |
| diphenyl | 14.6 | 5.95 | 2.88 | 2.02 | 0.40 | 2.26 | 1114 | 1276 |
| epidote | 202 | 212 | 45 | −14.3 | 0 | 43.2 | 3400 | 3409 |
| gypsum | 50.2 | 94.5 | 28.2 | −7.5 | −11.0 | 32.4 | 2310 | 3011 |
| hornblende | 192 | 116 | 61 | 10 | 4 | 31.8 | 3120 | 3049 |
| microcline | 122 | 66 | 26 | −13 | −3 | 23.8 | 2561 | 2816 |
| oligoclase | 124 | 81 | 54 | −7 | 16 | 27.4 | 2638 | 2413 |
| tartaric acid | 46.5 | 93 | 36.7 | −0.4 | −12.0 | 8.20 | 1760 | 1756 |
| tin fluoride | 33.6 | 47.9 | 5.3 | 6.5 | −5.1 | 12.9 | 4875 | 1339 |

nents, because the boundary conditions are simply written, using Eqs. (7), (5), and (13), as

$$t_i(0)=t_i(\infty)=0 \quad (i=1,2). \tag{22}$$

This claim is further justified in the next section, where the secular equation is quickly derived.

## VI. SECULAR EQUATION

Now the method of first integrals is applied to the system (19). Mozhaev[11] defined the following inner product,

$$(f,\phi)=\int_0^\infty (f\overline{\phi}+\overline{f}\phi)\,dx_2, \tag{23}$$

and multiplying Eq. (19) by $i\overline{t_j}$ gives

$$\hat{\alpha}_{ik}D_{kj}+\hat{\beta}_{ik}E_{kj}+\hat{\gamma}_{ik}F_{kj}=0, \tag{24}$$

where the $2\times 2$ matrices **D**, **E**, **F**, are defined by

$$D_{kj}=(it_k'',t_j), \quad E_{kj}=(t_k'',t_j), \quad F_{kj}=(t_k,it_j). \tag{25}$$

By writing down $F_{kj}+F_{jk}$, it is easy to check that the matrix **F** is antisymmetric. Integrating directly $E_{kj}+E_{jk}$, and integrating $D_{kj}+D_{jk}$ by parts, and using the boundary conditions Eq. (22), it is found that the matrices **E** and **D** are also antisymmetric. So **D**, **E**, and **F** may be written in the form

$$\mathbf{D}=\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} \quad \mathbf{E}=\begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix} \quad \mathbf{F}=\begin{bmatrix} 0 & F \\ -F & 0 \end{bmatrix}, \tag{26}$$

and Eq. (24) yields the following system of three linearly independent equations for the three unknowns $D$, $E$, $F$,

$$\begin{aligned} &\hat{\alpha}_{11}D+\hat{\beta}_{11}E+\hat{\gamma}_{11}F=0,\\ &\hat{\alpha}_{12}D+\hat{\beta}_{12}E+\hat{\gamma}_{12}F=0,\\ &\hat{\alpha}_{22}D+\hat{\beta}_{22}E+\hat{\gamma}_{22}F=0. \end{aligned} \tag{27}$$

This homogeneous linear algebraic system yields nontrivial solutions for $D$, $E$, and $F$, only when its determinant is zero, which, accounting for the fact that $\hat{\alpha}_{12}=\hat{\beta}_{22}=0$, is equivalent to $\hat{\beta}_{12}(\hat{\alpha}_{11}\hat{\gamma}_{22}-\hat{\alpha}_{22}\hat{\gamma}_{11})=-\hat{\alpha}_{22}\hat{\beta}_{11}\hat{\gamma}_{12}$, or equivalently, using the expressions Eqs. (21) and multiplying by $X^3(\eta-X)^3$,

$$\begin{aligned} &[\eta-(1+r_2)X]\{(\eta-X)[(\eta-X)(s_{22}X-1)+r_6^2X]\\ &\quad+X^2[(\eta-X)s_{66}+r_2^2]\}\\ &\quad=2r_6X^2(\eta-X)[(\eta-X)s_{26}-r_2r_6]. \end{aligned} \tag{28}$$

Hence *the secular equation is obtained explicitly as the quartic Eq. (28) in* $X=\rho v^2$, with coefficients expressed in terms of the elastic stiffnesses through Eqs. (16).

For consistency, the orthorhombic case, where $c_{16}=c_{26}=c_{45}=0$, is now considered. In this case, the coefficients Eq. (16) reduce to

$$\begin{aligned} &r_6=0, \quad r_2=\frac{c_{12}}{c_{22}}, \quad s_{22}=\frac{1}{c_{66}},\\ &s_{26}=0, \quad s_{66}=\frac{1}{c_{22}}, \quad \eta=c_{11}-\frac{c_{12}^2}{c_{22}}, \end{aligned} \tag{29}$$

and the right hand-side of Eq. (28) is zero, while the left hand-side yields the equation

$$\begin{aligned} &[\eta-(1+r_2)X]\{(\eta-X)^2(s_{22}X-1)\\ &\quad+X^2[(\eta-X)s_{66}+r_2^2]\}=0. \end{aligned} \tag{30}$$

The nullity of the first factor in this equation corresponds to $\hat{\beta}_{12}=0$. Because for the orthorhombic case, $\hat{\alpha}_{12}=\hat{\gamma}_{12}=\hat{\beta}_{11}=\hat{\beta}_{22}=0$ also, the equations of motion (19) then decouple into

$$\hat{\alpha}_{11}t_1''+\hat{\gamma}_{11}t_1=0, \quad \hat{\alpha}_{22}t_2''+\hat{\gamma}_{22}t_2=0, \tag{31}$$

whose solutions satisfying the boundary conditions Eqs. (22) are the trivial ones. The nullity of the second factor in Eq. (30) corresponds to the well-studied[6,7,17] secular equation for surface waves in orthorhombic crystals,

$$\begin{aligned} &\frac{c_{22}}{c_{11}}\left(\frac{c_{11}c_{22}-c_{12}^2}{c_{22}c_{66}}-\frac{\rho v^2}{c_{66}}\right)^2\left(1-\frac{\rho v^2}{c_{66}}\right)\\ &\quad-\left(\frac{\rho v^2}{c_{66}}\right)^2\left(1-\frac{\rho v^2}{c_{11}}\right)=0. \end{aligned} \tag{32}$$

Finally, concrete examples are given (see Table I). In each considered case, the secular equation (28) has either two or four positive real roots, out of which only one corresponds to a subsonic wave. The elimination of the other roots is made by comparison with the speed of a homogeneous body wave propagating in the direction of the $x_1$ material axis. For this body wave, the functions $U_i(x_2)$, $t_i(x_2)$, $(i=1,2)$, are constant, and the equations of motion imply that the determinant of the $4\times4$ matrix in Eq. (14) is zero, condition from which the body wave speed can be found. Also, it is checked a posteriori that the value $X=\eta$ corresponds to the supersonic range, and so that the matrix $N_3+X\mathbf{1}$ is indeed invertible within the subsonic range. For instance, for tin fluoride, $\eta$ is of the order of $3\times10^7$, the secular equation (28) has the roots 1339, 2350, 2513, and 3403, and the slowest body wave in the $x_1$ direction travels at 1504 m $s^{-1}$; hence a subsonic surface wave travels in tin fluoride at 1339 m $s^{-1}$.

Barnett, Chadwick, and Lothe,[18] and Chadwick and Willson[9] considered surface waves propagating in monoclinic materials, and computed the surface wave speed $v$ in two steps, first by solving numerically a bicubic, then by substituting the result into another equation of which $v$ is the only zero. These authors studied surface wave propagation for every value of the angle $\alpha$ between the reference plane and the plane of material symmetry. Numerical values for $v$ are only given in the cases of aegirite-augite, diallage, gypsum, and microcline, and at $\alpha=0$, these results are in agreement with those presented in Table I. Sources of experimental data and extensive discussions on limiting speeds, existence of secluded supersonic surface waves, rotation of the reference plane with respect to the plane of material symmetry, etc., can be found in these articles and in references therein.

## VII. CONCLUDING REMARKS

Surface wave motion in monoclinic crystals with plane of symmetry at $x_3=0$ turned out to correspond to plane strain and plane stress motion (Sec. IV). Thanks to this, the equations of of motion yielded a system of only two differential equations for the tractions (Sec. V). Once the method of first integrals was applied, a homogeneous system of three linearly independent equations for three unknowns was obtained (Sec. VI). Had the motion not corresponded to plane stress, then the same procedure would have given a system of 18 equations for 18 unknowns, when the equations of motion are written for the displacement components,[11] or a system of 9 equations for 9 unknowns, when the equations of motion are written for the traction components as in the present paper. However, these equations are not linearly independent, and the secular equation cannot be obtained in this manner. Hence, it ought to be stressed again that the method presented in the paper is not a general method for a surface wave traveling in arbitrary direction in an anisotropic crystal, but was limited to the study of a surface wave propagating in the $x_1$-direction of a monoclinic crystal with plane of symmetry at $x_3=0$, with attenuation in the $x_2$-direction.

Nevertheless, some plane strain problems remain open and it is hoped that the method exposed in this paper might help solve them analytically. Also, beyond mathematical satisfaction, the derivation of an explicit secular equation provides a basis for a possible nonlinear perturbative analysis.


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