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Circularly polarized plane waves in a deformed Hadamard material

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Abstract

Small-amplitude inhomogeneous plane waves propagating in any direction in a homogeneously deformed Hadamard material are considered. Conditions for circular polarization are established. The analysis relies on the use of complex vectors (or bivectors) to describe the slowness and the polarization of the waves.

Generally, homogeneous circularly polarized plane waves may propagate in only two directions, the directions of the acoustic axes, in a homogeneously deformed Hadamard material. For inhomogeneous circularly polarized plane waves, the number of possibilities is far greater. They include an infinity of 'transverse waves', as well as 'longitudinal waves', and the superposition of transverse waves and longitudinal waves, where 'transverse' and 'longitudinal' are used in the bivector sense.

Each and every possibility of circular polarization is examined in turn, and explicit examples of solutions are given in every case.

1. Introduction

When a homogeneous, isotropic, compressible elastic body is maintained in a state of finite homogeneous static deformation, longitudinal waves are, in general, only possible when the propagation direction is along a principal axis of strain. However, Hadamard [1] introduced a remarkable class of elastic materials characterized by the property that *infinitesimal* longitudinal waves may propagate in every direction, irrespective of the basic static finite homogeneous deformation. Many studies of the properties of Hadamard materials have been made [2–4]. Among these we mention the result of Boulanger et al. [5], who showed that there are only two directions, $\mathbf{n}^+$ and $\mathbf{n}^-$, called 'acoustic axes', along which *finite-amplitude* circularly polarized transverse waves may propagate. The acoustic axes $\mathbf{n}^\pm$ are the directions along the normals to the planes of central circular sections of the ellipsoid $\mathbf{x} \cdot \mathbb{B}^{-1}\mathbf{x} = 1$, where the left Cauchy–Green tensor associated with the basic deformation is denoted by $\mathbb{B}$. The acoustic axes are determined solely by the basic static deformation and are independent of the choice of material constants and of the function which occurs in the strain-energy function describing the Hadamard material.

Here, consideration is restricted to the propagation of *infinitesimal* plane waves in a Hadamard material maintained in a state of finite static homogeneous deformation, and the primary emphasis of this paper is on *inhomogeneous* plane waves. For these waves, the incremental displacement $\mathbf{u}$ is of the form $\mathbf{u} = \mathbf{A} \exp \mathrm{i}\omega(N\mathbf{C} \cdot \mathbf{x} - t)$, where $\mathbf{A}$

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and $\mathbf{C}$ are the amplitude and propagation bivectors, respectively, ω the real frequency, and N the complex scalar slowness. The propagation bivector $\mathbf{C} = \mathbf{C}^+ + \mathrm{i}\mathbf{C}^-$ is prescribed with $\mathbf{C}^+ \neq \mathbf{0}$, $\mathbf{C}^- \neq \mathbf{0}$, and $\mathbf{A}$ and N are sought such that $\mathbf{u}$ satisfies the equations of motion [6]. This is equivalent to finding the eigenvalues ρN^{-2} and eigenbivectors $\mathbf{A}$ of the acoustical tensor $\mathbb{Q}(\mathbf{C})$: $\mathbb{Q}(\mathbf{C})\mathbf{A} = \rho N^{-2}\mathbf{A}$. Our special consideration is the determination of circularly polarized inhomogeneous plane wave solutions. For these waves, the amplitude bivector $\mathbf{A}$ must be isotropic: $\mathbf{A} \cdot \mathbf{A} = 0$. We find that there are two distinct sets of solutions, according as to whether or not $\mathbf{C}$ is chosen to be isotropic.

In the case where $\mathbf{C}$ is chosen non-isotropic ($\mathbf{C} \cdot \mathbf{C} \neq 0$), the 'projection tensor' $\Pi = \mathbf{1} - \{\mathbf{C} \otimes \mathbf{C}/(\mathbf{C} \cdot \mathbf{C})\}$, may be introduced. For this, $\mathbf{\Pi}^2 = \mathbf{\Pi}$, $\mathbf{\Pi C} = \mathbf{0}$. If $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- \neq 0$, then $\mathbf{\Pi n}^+$ is an isotropic amplitude bivector. The orthogonal projection of the directional ellipse of $\mathbf{C}$, onto the plane of central circular section of the $\mathbb{B}^{-1}$-ellipsoid with normal $\mathbf{n}^+$, is a circle. There is an infinity of such choices of $\mathbf{C}$ and, therefore, an infinity of circularly polarized inhomogeneous plane waves (similar comments apply when $\mathbf{C} \cdot \mathbf{C} \neq 0$, $\mathbf{n}^- \cdot \Pi\mathbf{n}^- = 0$, $\mathbf{n}^+ \cdot \Pi\mathbf{n}^+ \neq 0$).

There are only two bivectors $\mathbf{C}$ satisfying $\mathbf{C} \cdot \mathbf{C} \neq 0$, $\mathbf{n}^+\mathbf{\Pi n}^+ = 0$, $\mathbf{n}^-\mathbf{\Pi n}^- = 0$. Then $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are both isotropic amplitude bivectors with the same eigenvalue. There are *two* circularly polarized inhomogeneous plane waves in this case.

Finally, if the orthogonal projections of the directional ellipse of $\mathbf{C}$ upon either plane of central circular section of the $\mathbb{B}^{-1}$-ellipsoid is not a circle, $\mathbf{n}^\pm \cdot \Pi\mathbf{n}^\pm \neq 0$, then the tensor $\mathbf{\Pi}\mathbb{B}^{-1}\mathbf{\Pi}$ admits two orthogonal non-isotropic eigenbivectors $\mathbf{A}^+$, $\mathbf{A}^-$, with different eigenvalues. In this case, if the directional ellipse of $\mathbf{C}$ is chosen to be similar and similarly situated to a (non-circular) central elliptical section of a certain $\mathbb{M}$-ellipsoid, $\mathbf{C} \cdot \mathbb{M}\mathbf{C} = 0$, where $\mathbb{M}$ depends upon the finite deformation, then there is a corresponding isotropic amplitude bivector $\mathbf{A}$ which is parallel to either $\{\mathbf{C}/(\mathbf{C} \cdot \mathbf{C})^{1/2}\} \pm \mathrm{i}\mathbf{A}^+$ or $\{\mathbf{C}/(\mathbf{C} \cdot \mathbf{C})^{1/2}\} \pm \mathrm{i}\mathbf{A}^-$. There is a corresponding infinity of inhomogeneous circularly polarized waves.

Turning now to the set of solutions for which $\mathbf{C}$ is isotropic, $\mathbf{C} \cdot \mathbf{C} = 0$, it is seen that if $\mathbf{C}$ is chosen isotropic and also satisfies $\mathbf{C} \cdot \mathbb{B}^{-1}\mathbf{C} = 0$, so that the circle of $\mathbf{C}$ lies in either plane of central circular section of the $\mathbb{B}^{-1}$-ellipsoid, then the corresponding amplitude bivectors $\mathbf{A}$ are parallel to $\mathbf{C}$. There are two such waves.

If $\mathbf{C}$ is isotropic, $\mathbf{C} \cdot \mathbf{C} = 0$, and if $\mathbf{C} \cdot \mathbb{B}^{-1}\mathbf{C} \neq 0$, $\mathbf{C} \cdot \mathbf{\Phi C} = 0$, where $\mathbf{\Phi}$ is a certain real symmetric tensor which depends only upon the basic deformation, then the circle of $\mathbf{C}$ lies in a plane of central circular section of the ellipsoid associated with $\mathbf{\Phi}$. Corresponding to either choice of $\mathbf{C}$, there are two linearly independent isotropic eigenbivectors of $\mathbb{Q}(\mathbf{C})$, each with the same eigenvalue. Thus, with either choice of $\mathbf{C}$, there are *two* circularly polarized waves which may propagate.

Finally, for isotropic $\mathbf{C}$, if $\mathbf{C} \cdot \mathbb{B}^{-1}\mathbf{C} \neq 0$, $\mathbf{C} \cdot \mathbf{\Phi C} \neq 0$, all isotropic amplitude eigenbivectors are parallel to $\mathbf{C}$. There is an infinity of such choices of $\mathbf{C}$ and, therefore, also of circularly polarized plane waves.

Examples are presented for every type of solution. The paper is organized as follows.

In Section 2, we recall the equations governing the behavior of a Hadamard material subjected to a finite static homogeneous deformation. For the constitutive equation, we follow Boulanger et al. [5] and choose to express the Cauchy stress in terms of $\mathbf{1}$, $\mathbb{B}$, and $\mathbb{B}^{-1}$, where $\mathbb{B}$ is the left Cauchy–Green strain tensor, rather than in terms of $\mathbf{1}$, $\mathbb{B}$, and $\mathbb{B}^2$ [3,4]. Then we assume that an infinite body of Hadamard material is subjected to a finite static pure homogeneous deformation. We also introduce the acoustic axes in the deformed state, which play an important role in the study of elastic waves.

Then in Section 3, we consider the propagation of small-amplitude inhomogeneous plane waves in a deformed body of Hadamard material. The waves are of complex exponential type and their associated amplitude and slowness are described through the use of bivectors [6]. Explicitly, the perturbation is the real part of $\mathbf{A} \exp \mathrm{i}\omega N(\mathbf{C} \cdot \mathbf{x} - t)$, where ω is the real frequency of the wave, and $\mathbf{A}$, $\mathbf{C}$, and N are complex quantities called the 'amplitude bivector', 'propagation bivector', and 'complex scalar slowness', respectively. Incremental strain, strain invariants, and stress are computed, leading to the derivation of the equations of motion and the acoustical tensor.

Next, we seek circularly polarized solutions, which correspond [6] to the isotropy of $\mathbf{A}$ (i.e. $\mathbf{A} \cdot \mathbf{A} = 0$) or equivalently to a double eigenvalue of the acoustical tensor.

The case of circularly polarized waves with a non-isotropic propagation bivector $\mathbf{C}$ (i.e. $\mathbf{C} \cdot \mathbf{C} \neq 0$) is treated in Section 4. We prove that the existence of such waves is determined by a condition linking $\mathbf{C}$ to some tensors which depend only on the finite static deformation. Transverse ($\mathbf{A} \cdot \mathbf{C} = 0$) waves are found, as well as waves which can be decomposed into the superposition of a transverse ($\mathbf{A} \cdot \mathbf{C} = 0$) wave and a longitudinal ($\mathbf{A} \wedge \mathbf{C} = \mathbf{0}$) wave.

In Section 5, we consider circularly polarized inhomogeneous plane waves with an isotropic bivector $\mathbf{C}$ (i.e. $\mathbf{C} \cdot \mathbf{C} = 0$). We show that longitudinal waves can propagate. We also find all other circularly polarized waves.

Finally, in Section 6, we specialize our results to the case of circularly polarized *homogeneous* plane waves, when $\mathbf{C}$ is a real unit vector in the direction of propagation. The result established by Boulanger et al. [5] for finite-amplitude plane waves is recovered: circular polarization for homogeneous waves occurs only in the directions orthogonal to either of the planes of central circular sections of the ellipsoid $\mathbf{x} \cdot \mathbb{B}^{-1}\mathbf{x} = 1$, where $\mathbb{B}$ is the left Cauchy–Green strain tensor of the finite static deformation.

2. Basic equations

2.1. Hadamard materials

We consider homogeneous isotropic hyperelastic materials of the Hadamard type. These are characterized by a strain-energy density Σ, measured per unit volume of the undeformed state, given by [2]

$$2\Sigma = aII + bI + f(III), \tag{2.1}$$

where a, b are two material constants. Also, $f(III)$ is a material function, and I, II, III are principal invariants of the left Cauchy–Green tensor $\mathbf{B}$

$$I = \operatorname{tr}\mathbf{B}, \qquad 2II = I^2 - \operatorname{tr}(\mathbf{B}^2), \qquad III = \det \mathbf{B}, \tag{2.2}$$

with $\mathbf{B}$ given by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \qquad B_{ij} = \left(\frac{\partial x_i}{\partial X_A}\right)\left(\frac{\partial x_j}{\partial X_A}\right), \qquad F_{iA} = \left(\frac{\partial x_i}{\partial X_A}\right), \tag{2.3}$$

where x_i are the coordinates at time t of a particle whose coordinates are X_A in the undeformed state. The deformation gradient $\mathbf{F}$ is such that $\det \mathbf{F} > 0$.

The constitutive equation for the Cauchy stress $\mathbf{t}$ for a Hadamard material is [5]:

$$\mathbf{t} = [aII\, III^{-1/2} + g(III^{1/2})]\mathbf{1} + b\mathbf{B} - aIII\,\mathbf{B}^{-1}, \tag{2.4}$$

where the function g is defined by

$$g = g(III^{1/2}) = III^{1/2} f'(III). \tag{2.5}$$

We exclude consideration of the special case where $a = 0$, when the material is a 'restricted Hadamard material' [7–10]. Then, assuming $a \neq 0$, it may be shown [4,5] that in order for the strong ellipticity conditions to hold, the following inequalities must be valid:

$$a > 0, \qquad b \geq 0, \qquad g' \geq 0. \tag{2.6}$$

It is assumed throughout that these conditions are satisfied.

We also assume that the body of Hadamard material is free of stress in the undeformed state. Thus $\mathbf{t} = \mathbf{0}$ when $\mathbf{B} = \mathbf{1}$. Then we must have [5]:

$$f'(1) = -(a + 2b). \tag{2.7}$$

The equations of motion, in the absence of body forces, are

$$\rho\ddot{\mathbf{x}} = \operatorname{div}\mathbf{t}, \qquad \rho\ddot{x}_i = \frac{\partial t_{ij}}{\partial x_j}, \tag{2.8}$$

where $\ddot{x}_i$ are the acceleration components. Also, ρ is the current mass density, related to the density ρ_0 of the material in the undeformed state through

$$\rho = III^{-1/2}\rho_0. \tag{2.9}$$

2.2. Homogeneously deformed Hadamard material

Now, we assume that a Hadamard material undergoes a finite pure homogeneous static deformation, bringing a particle initially at $\mathbf{X}$ in the undeformed state to $\mathbf{x}$ in the deformed state. Let $(\mathrm{O}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ be a rectangular Cartesian coordinate system defined by an origin O and the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ along the three directions of principal stretches. Then the finite homogeneous deformation is given by

$$\mathbf{x} = \lambda_1 X\mathbf{i} + \lambda_2 Y\mathbf{j} + \lambda_3 Z\mathbf{k}, \tag{2.10}$$

where λ_α, $\alpha = 1, 2, 3$ are the stretch ratios in each principal direction. Throughout the paper, it is assumed that these ratios are distinct and ordered as

$$\lambda_1 > \lambda_2 > \lambda_3. \tag{2.11}$$

Let $\mathbb{F}$ and $\mathbb{B}$ be the deformation gradient and left Cauchy–Green strain tensor corresponding to this deformation. Then

$$\mathbb{F} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad \mathbb{B} = \operatorname{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2), \tag{2.12}$$

and the corresponding strain invariants I, II, III are

$$I = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad II = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \qquad III = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.13}$$

In this deformed state, two specific directions play an important role with respect to the propagation of finite-amplitude homogeneous plane waves. They are the so-called acoustic axes, whose directions are the only ones along which finite-amplitude circularly polarized waves may propagate. It has been proved [5] for a Hadamard material that the acoustic axes are the normals to the planes of central circular sections of the $\mathbb{B}^{-1}$-ellipsoid ($\mathbf{x} \cdot \mathbb{B}^{-1}\mathbf{x} = 1$), which are along the unit vectors $\mathbf{n}^{\pm}$ defined by

$$\begin{gathered}\mathbf{n}^{\pm} = \alpha\mathbf{i} \pm \gamma\mathbf{j}, \qquad \alpha^2 + \gamma^2 = 1, \qquad \alpha = \sqrt{\frac{\lambda_2^{-2} - \lambda_1^{-2}}{\lambda_3^{-2} - \lambda_1^{-2}}}, \qquad \gamma = \sqrt{\frac{\lambda_3^{-2} - \lambda_2^{-2}}{\lambda_3^{-2} - \lambda_1^{-2}}}, \\ \alpha^2\lambda_3^{-2} + \gamma^2\lambda_1^{-2} = \lambda_2^{-2}, \qquad \gamma^2\lambda_1^2 + \alpha^2\lambda_3^2 - \lambda_2^2 = \alpha^2\gamma^2 III(\lambda_3^{-2} - \lambda_1^{-2})^2.\end{gathered} \tag{2.14}$$

These unit vectors $\mathbf{n}^{\pm}$ also appear in the Hamilton cyclic decomposition of the $\mathbb{B}^{-1}$ tensor [11]:

$$\mathbb{B}^{-1} = \lambda_2^{-2}\mathbf{1} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})[\mathbf{n}^+ \otimes \mathbf{n}^- + \mathbf{n}^- \otimes \mathbf{n}^+], \tag{2.15}$$

where $\otimes$ denotes the dyadic product.

Finally, the constant Cauchy stress tensor $\mathbb{T}$ necessary to maintain the finite homogeneous deformation (2.10) is given by [5]:

$$\mathbb{T} = [aII\,III^{-1/2} + g(III^{1/2})]\mathbf{1} + bIII^{-1/2}\mathbb{B} - aIII^{1/2}\mathbb{B}^{-1}. \tag{2.16}$$

3. Small-amplitude plane waves in a deformed Hadamard material

Now we consider the propagation of an infinitesimal plane wave of complex exponential type in a Hadamard material held in a state of static finite homogeneous deformation. The emphasis is on inhomogeneous plane waves. We derive the equations of motion and the corresponding acoustical tensor.

3.1. Plane waves of complex exponential type

We assume that a plane wave of complex exponential type is superposed upon the finite static deformation described in Section 2.2. This motion, which brings a particle from $\mathbf{x}$, given by (2.10), to $\bar{\mathbf{x}}$ in the current configuration of the material, is written as [6]:

$$\bar{\mathbf{x}} = \mathbf{x} + \tfrac{1}{2}\epsilon\{\mathbf{A}\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}. \tag{3.1}$$

Here, ϵ is a small parameter, such that terms of order ϵ^2 or higher may be neglected in comparison with first order terms, $\mathbf{A}$ (amplitude) and $\mathbf{C}$ (propagation) are complex vectors (or 'bivectors' [6]), ω the real frequency, N the complex scalar slowness, and 'c.c.' the complex conjugate.

An ellipse may be associated with a bivector [6]. Thus, if the bivector $\mathbf{D}$ has real and imaginary parts $\mathbf{D}^+$ and $\mathbf{D}^-$, so that $\mathbf{D} = \mathbf{D}^+ + \mathrm{i}\mathbf{D}^-$, then the equation of the corresponding ellipse is $\mathbf{r} = \mathbf{D}^+\cos\theta + \mathbf{D}^-\sin\theta$, $0 \le \theta \le 2\pi$. The ellipse is a circle if $\mathbf{D}\cdot\mathbf{D} = 0$, and degenerates to a line segment if $\mathbf{D}\wedge\bar{\mathbf{D}} = \mathbf{0}$.

In the case where the amplitude bivector $\mathbf{A}$ is such that

$$\mathbf{A}\cdot\mathbf{A} = 0, \tag{3.2}$$

the wave described by (3.1) is circularly polarized and the amplitude bivector $\mathbf{A}$ is said to be 'isotropic'. If $\mathbf{A}\wedge\bar{\mathbf{A}} = \mathbf{0}$, the wave is linearly polarized, and $\mathbf{A}$ has 'a real direction'.

In general, the propagation bivector $\mathbf{C}$ may be written as [11]:

$$\mathbf{C} = m\hat{\mathbf{m}} + \mathrm{i}\hat{\mathbf{n}}, \tag{3.3}$$

where m is a real number ($m \ge 1$) and $\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$ are real orthogonal unit vectors. By suitable choices of m, $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$, all possible propagation bivectors $\mathbf{C}$ are determined [6].

3.2. Strain increments

Let the deformation gradient corresponding to the motion (3.1) be $\bar{\mathbb{F}}$. It is given by

$$\bar{\mathbb{F}} = \frac{\partial\bar{\mathbf{x}}}{\partial\mathbf{X}} = [\mathbf{1} + \frac{1}{2}\epsilon\{\mathrm{i}\omega N\mathbf{A}\otimes\mathbf{C}\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}]\mathbb{F}. \tag{3.4}$$

The left Cauchy–Green tensor $\bar{\mathbb{B}} = \bar{\mathbb{F}}\bar{\mathbb{F}}^{\mathrm{T}}$ and its inverse $\bar{\mathbb{B}}^{-1}$ are, up to order ϵ, given by

$$\bar{\mathbb{B}} = \mathbb{B} + \tfrac{1}{2}\epsilon\{\mathrm{i}\omega N[\mathbf{A}\otimes\mathbb{B}\mathbf{C} + \mathbb{B}\mathbf{C}\otimes\mathbf{A}]\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \tag{3.5}$$

$$\bar{\mathbb{B}}^{-1} = \mathbb{B}^{-1} - \tfrac{1}{2}\epsilon\{\mathrm{i}\omega N[\mathbf{C}\otimes\mathbb{B}^{-1}\mathbf{A} + \mathbb{B}^{-1}\mathbf{A}\otimes\mathbf{C}]\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}. \tag{3.6}$$

The corresponding strain invariants $\bar{I}$, $\overline{II}$, $\overline{III}$ are given by

$$\bar{I} = I + \epsilon\{\mathrm{i}\omega N(\mathbf{A}\cdot\mathbb{B}\mathbf{C})\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \tag{3.7}$$

$$\overline{II} = II + \epsilon\{\mathrm{i}\omega N[II(\mathbf{A}\cdot\mathbf{C}) - III(\mathbf{A}\cdot\mathbb{B}^{-1}\mathbf{C})]\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \tag{3.8}$$

$$\overline{III} = III + \epsilon III\{\mathrm{i}\omega N(\mathbf{A}\cdot\mathbf{C})\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}. \tag{3.9}$$

Finally, the mass density $\bar{\rho}$ in the current configuration is

$$\bar{\rho} = III^{-1/2}\rho(1 - \tfrac{1}{2}\epsilon\{\mathrm{i}\omega N(\mathbf{A}\cdot\mathbf{C})\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}). \tag{3.10}$$

3.3. Equations of motion and acoustical tensor

The Cauchy stress $\bar{\mathbb{T}}$ necessary to support the motion (3.1) is [5]:

$$\bar{\mathbb{T}} = [a\overline{II}\,\overline{III}^{-1/2} + g(\overline{III}^{1/2})]\mathbf{1} + b\overline{III}^{-1/2}\bar{\mathbb{B}} - a\overline{III}^{1/2}\bar{\mathbb{B}}^{-1}. \tag{3.11}$$

Using (3.5)–(3.11), we find that

$$\bar{\mathbb{T}} = \mathbb{T} + \tfrac{1}{2}\epsilon\{\mathrm{i}\omega N\hat{\mathbb{T}}\,\mathrm{e}^{\mathrm{i}\omega(N\mathbf{C}\cdot\mathbf{x}-t)} + \text{c.c.}\}, \tag{3.12}$$

where $\hat{\mathbb{T}}$ is given by

$$\begin{aligned}\hat{\mathbb{T}} = {}&\{(\mathbf{A}\cdot\mathbf{C})g' + a[(\operatorname{tr}\mathbb{B}^{-1})(\mathbf{A}\cdot\mathbf{C}) - 2(\mathbf{A}\cdot\mathbb{B}^{-1}\mathbf{C})]\}III^{1/2}\mathbf{1} - b(\mathbf{A}\cdot\mathbf{C})III^{-1/2}\mathbb{B}\\ &+bIII^{-1/2}[\mathbf{A}\otimes\mathbb{B}\mathbf{C} + \mathbb{B}\mathbf{C}\otimes\mathbf{A}] - a(\mathbf{A}\cdot\mathbf{C})III^{1/2}\mathbb{B}^{-1} + aIII^{1/2}[\mathbf{C}\otimes\mathbb{B}^{-1}\mathbf{A} + \mathbb{B}^{-1}\mathbf{A}\otimes\mathbf{C}].\end{aligned} \tag{3.13}$$

Now, in the absence of body forces, the equations of motion (2.8), written for the motion (3.1), are: $\operatorname{div}\bar{\mathbb{T}} = \bar{\rho}\ddot{\bar{\mathbf{x}}}$. In our context, they yield

$$-\omega^2N^2\hat{\mathbb{T}}\cdot\mathbf{C} = -\rho\omega^2III^{-1/2}\mathbf{A} \quad \text{or} \quad \mathbb{Q}(\mathbf{C})\mathbf{A} = \rho N^{-2}\mathbf{A}, \tag{3.14}$$

where the *acoustical tensor* $\mathbb{Q}(\mathbf{C})$ is given by

$$\mathbb{Q}(\mathbf{C}) = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C})\mathbf{1} + (IIIg' + aII)\mathbf{C}\otimes\mathbf{C} + aIII[(\mathbf{C}\cdot\mathbf{C})\mathbb{B}^{-1} - \mathbf{C}\otimes\mathbb{B}^{-1}\mathbf{C} - \mathbb{B}^{-1}\mathbf{C}\otimes\mathbf{C}]. \tag{3.15}$$

By inspection of the form of $\mathbb{Q}(\mathbf{C})$, we deduce two facts about the acoustical tensor.

First, the acoustical tensor is a complex symmetric tensor

$$\mathbb{Q} = \mathbb{Q}^{\mathrm{T}}, \tag{3.16}$$

and, therefore, eigenbivectors of $\mathbb{Q}(\mathbf{C})$ corresponding to distinct eigenvalues will be orthogonal to each other (e.g. [11]).

Second, the propagation bivector $\mathbf{C}$ is an eigenbivector of $\mathbb{Q}(\mathbf{C})$ with eigenvalue $\rho N_{\parallel}^{-2}$ (say) given by

$$\rho N_{\parallel}^{-2} = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) + (IIIg' + aII)(\mathbf{C}\cdot\mathbf{C}) - aIII(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}). \tag{3.17}$$

We now seek small-amplitude circularly polarized inhomogeneous plane wave solutions to the equations of motion in a deformed Hadamard material. These solutions correspond to a double root of the acoustical tensor, or equivalently [6], to an isotropic amplitude eigenbivector $\mathbf{A}$ for $\mathbb{Q}(\mathbf{C})$. We follow the usual [12,13] procedure for inhomogeneous solutions of complex exponential type, which consists in separating the cases where the propagation bivector $\mathbf{C}$ is non-isotropic ($\mathbf{C}\cdot\mathbf{C}\neq 0$), from the cases where it is isotropic ($\mathbf{C}\cdot\mathbf{C} = 0$).

4. Circularly polarized inhomogeneous plane waves with a non-isotropic propagation bivector C

Here we determine all possible circularly polarized solutions of complex exponential type with a non-isotropic bivector $\mathbf{C}$: $\mathbf{C}\cdot\mathbf{C}\neq 0$. It is seen that these waves can be constructed not only as 'transverse' waves—in the sense that $\mathbf{A}\cdot\mathbf{C} = 0$, but also as the superposition of a transverse wave and 'longitudinal' wave—in the sense that $\mathbf{A}\wedge\mathbf{C} = \mathbf{0}$. Explicit solutions are presented.

Assuming that the ellipse of $\mathbf{C}$ is not a circle, so that $\mathbf{C}\cdot\mathbf{C}\neq 0$, the acoustical tensor $\mathbb{Q}(\mathbf{C})$ may be written

$$\mathbb{Q}(\mathbf{C}) = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C})\mathbf{1} + [IIIg' + aII - aIII\mathbf{C}^*\cdot\mathbb{B}^{-1}\mathbf{C}^*]\mathbf{C}\otimes\mathbf{C} + aIII(\mathbf{C}\cdot\mathbf{C})\boldsymbol{\chi}, \tag{4.1}$$

where

$$\mathbf{C}^* = \frac{\mathbf{C}}{(\mathbf{C}\cdot\mathbf{C})^{1/2}}, \qquad \boldsymbol{\chi} = \boldsymbol{\Pi}\mathbb{B}^{-1}\boldsymbol{\Pi}, \qquad \boldsymbol{\Pi} = \mathbf{1} - \mathbf{C}^*\otimes\mathbf{C}^*, \qquad \boldsymbol{\Pi}^2 = \boldsymbol{\Pi}, \qquad \boldsymbol{\Pi}\mathbf{C} = \mathbf{0}. \tag{4.2}$$

Also $\mathbf{\Pi A} = \mathbf{A}$ when $\mathbf{A} \cdot \mathbf{C} = 0$. The tensor $\mathbf{\Pi}$ is called the 'complex projection operator'. Detailed properties of $\mathbf{\Pi}$ are given in Appendix A. In particular, properties of $\boldsymbol{\chi}$ are presented there.

We recall that $\mathbb{Q}(\mathbf{C})$ has eigenbivector $\mathbf{C}$ with eigenvalue $\rho N_{\parallel}^{-2}$ given by (3.17):

$$\mathbb{Q}(\mathbf{C})\mathbf{C} = \rho N_{\parallel}^{-2}\mathbf{C}. \tag{4.3}$$

We wish to determine the other eigenbivectors. Four cases have to be considered (see Appendix A):

Case (i): $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ \neq 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- \neq 0$.
Case (iia): $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- \neq 0$.
Case (iib): $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ \neq 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$.
Case (iii): $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$.

4.1. Case (i): superposition of transverse and longitudinal waves

Here, $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ \neq 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- \neq 0$. It follows that the orthogonal projection of the ellipse of $\mathbf{C}$ upon either plane of central circular section of the $\mathbb{B}^{-1}$-ellipsoid is not a circle. It will be seen that the acoustical tensor has $\mathbf{C}$ as an eigenbivector, so that the corresponding wave may be called longitudinal, and two further distinct orthogonal eigenbivectors both orthogonal to $\mathbf{C}$, so that the corresponding waves may be called transverse. The transverse wave slownesses are distinct, so that for a given $\mathbf{C}$, if two wave slownesses are to be equal, then this possibility will only occur if the wave slowness for one of the transverse waves is equal to the wave slowness of the longitudinal wave. It will be seen that there is an infinite number of possible choices of $\mathbf{C}$ for which this is so. Then the corresponding waves are circularly polarized. Any possible $\mathbf{C}$ has an ellipse which is similar and similarly situated to the ellipse in which the plane of $\mathbf{C}$ cuts a certain $\mathbb{M}$-ellipsoid, with the two exceptions when the plane of $\mathbf{C}$ coincides with a plane of central circular section of the $\mathbb{M}$-ellipsoid.

It may be shown (see Appendix A) that $\boldsymbol{\chi}$ has unit orthogonal eigenbivectors $\mathbf{A}^{\pm}$, given by

$$\mathbf{A}^{\pm} = \frac{\mathbf{K}^+ \pm \mathbf{K}^-}{[2(1+\mathbf{K}^+ \cdot \mathbf{K}^-)]^{1/2}}, \qquad \mathbf{K}^{\pm} = \frac{\mathbf{\Pi n}^{\pm}}{(\mathbf{n}^{\pm} \cdot \mathbf{\Pi n}^{\pm})^{1/2}}, \tag{4.4}$$

and corresponding distinct eigenvalues $\delta^{\pm}$, given by

$$\delta^{\pm} = \lambda_2^{-2} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})[(\mathbf{n}^+ \cdot \mathbf{\Pi n}^+)(\mathbf{n}^- \cdot \mathbf{\Pi n}^-)]^{1/2}(\mathbf{K}^+ \cdot \mathbf{K}^- \pm 1). \tag{4.5}$$

Also, $\mathbf{A}^{\pm}$ are orthogonal to $\mathbf{C}$, the other eigenbivector of $\boldsymbol{\chi}$, with corresponding eigenvalue zero.

So the eigenbivectors of $\mathbb{Q}(\mathbf{C})$ are: $\mathbf{C}$, with eigenvalue $\rho N_{\parallel}^{-2}$, and $\mathbf{A}^{\pm}$, with corresponding distinct eigenvalues $b\mathbf{C} \cdot \mathbb{B}\mathbf{C} + aIII(\mathbf{C} \cdot \mathbf{C})\delta^{\pm}$. Because the eigenvalues corresponding to $\mathbf{A}^+$ and $\mathbf{A}^-$ are distinct, the only way in which an isotropic eigenbivector of $\mathbb{Q}(\mathbf{C})$ will arise is when the eigenvalue $\rho N_{\parallel}^{-2}$ is equal to one of the eigenvalues corresponding to either $\mathbf{A}^+$ or to $\mathbf{A}^-$. Thus, for an isotropic eigenbivector we need $\mathbf{C}$ to satisfy

$$[\rho N_{\parallel}^{-2} - b\mathbf{C} \cdot \mathbb{B}\mathbf{C} - aIII(\mathbf{C} \cdot \mathbf{C})\delta^+][\rho N_{\parallel}^{-2} - b\mathbf{C} \cdot \mathbb{B}\mathbf{C} - aIII(\mathbf{C} \cdot \mathbf{C})\delta^-] = 0. \tag{4.6}$$

Using (3.17) and (A.22), this equation may be written

$$\mathbf{C} \cdot \mathbb{M}\mathbf{C} = 0, \tag{4.7}$$

where $\mathbb{M}$ is the tensor defined by

$$\mathbb{M} = (g')^2 III\mathbf{1} + ag'[II\mathbf{1} - III\mathbb{B}^{-1}] + a^2\mathbb{B}. \tag{4.8}$$

We note that $\mathbb{M}$ is coaxial with $\mathbb{B}$ and has eigenvalues μ_α^2, given by

$$\mu_\alpha^2 = (g')^2 III + ag'(II - III\lambda_\alpha^{-2}) + a^2\lambda_\alpha^2, \quad \alpha = 1, 2, 3. \tag{4.9}$$

Using (2.6) and (2.11), we find that

$$\mu_1^2 > \mu_2^2 > \mu_3^2 > 0. \tag{4.10}$$

Thus, $\mathbb{M}$ is a positive definite tensor, determined by the finite static deformation, and the associated quadric $\mathbf{x}\cdot\mathbb{M}\mathbf{x} = 1$ is an ellipsoid.

So, now if $\mathbf{C}$ is chosen to satisfy (4.7), i.e. if the plane of the ellipse of $\mathbf{C}$ cuts the $\mathbb{M}$-ellipsoid in an ellipse which is similar and similarly situated to the ellipse of $\mathbf{C}$, then $\mathbb{Q}(\mathbf{C})$ has a double eigenvalue, $\rho N_{\parallel}^{-2}$, corresponding to the eigenbivector $\mathbf{C}$, and also either to $\mathbf{A}^+$ or to $\mathbf{A}^-$. Consequently, all isotropic eigenbivectors $\mathbf{A}$ of the acoustical tensor $\mathbb{Q}(\mathbf{C})$ must be orthogonal to $\mathbf{A}^+$ (or $\mathbf{A}^-$), be such that $\mathbf{A}\cdot\mathbf{A} = 0$, and consequently, up to a scalar factor, be of the form

$$\mathbf{A} = \mathbf{C}^* + \mathrm{i}\mathbf{A}^- \quad \text{or} \quad \mathbf{C}^* + \mathrm{i}\mathbf{A}^+, \tag{4.11}$$

with corresponding eigenvalue $\rho N_{\parallel}^{-2}$. Of course, the amplitude bivector $\mathbf{A}$ is a combination of $\mathbf{C}^*$, which corresponds to a longitudinal wave in the sense that the amplitude bivector is 'parallel' to the propagation bivector $\mathbf{C}$, and of the amplitude $\mathbf{A}^-$, which corresponds to a transverse wave in the sense that the amplitude bivector $\mathbf{A}^-$ is 'orthogonal' to the propagation bivector $\mathbf{C}$.

There is an infinity of possible choices for $\mathbf{C}$ which satisfy (4.7). Any elliptical section of the $\mathbb{M}$-ellipsoid may be chosen for $\mathbf{C}$, apart from the two central circular sections—we recall that $\mathbf{C}$ may not be isotropic. These 'forbidden' isotropic $\mathbf{C}$, in the planes of central circular section of the $\mathbb{M}$-ellipsoid, are

$$\mathbf{C} = \frac{\gamma(a+\lambda_1^2 g')^{1/2}}{\lambda_1(a+\lambda_2^2 g')^{1/2}}\mathbf{i} \pm \mathrm{i}\mathbf{j} \mp \frac{\alpha(a+\lambda_3^2 g')^{1/2}}{\lambda_3(a+\lambda_1^2 g')^{1/2}}\mathbf{k}. \tag{4.12}$$

4.2. Example: superposition of transverse and longitudinal waves

Let $\mathbf{C}$ be written $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$. The condition (4.7) is $C_1^2\mu_1^2 + C_2^2\mu_2^2 + C_3^2\mu_3^2 = 0$. Recalling (4.10) we choose

$$\mathbf{C} = \mu_2\mathbf{i} + \mathrm{i}\mu_1\mathbf{j}. \tag{4.13}$$

Then

$$\mathbf{\Pi n}^{\pm} = -\frac{\alpha\mu_1^2}{\mu_2^2-\mu_1^2}\mathbf{i} \pm \gamma\mathbf{k} - \mathrm{i}\frac{\alpha\mu_1\mu_2}{\mu_2^2-\mu_1^2}\mathbf{j}. \tag{4.14}$$

It may be checked that $\mathbf{n}^+\cdot\mathbf{\Pi n}^+ = \mathbf{n}^-\cdot\mathbf{\Pi n}^-$, so that $\mathbf{A}^{\pm}$ are parallel to $\mathbf{\Pi n}^+ \pm \mathbf{\Pi n}^-$. Thus,

$$\mathbf{A}^+ = \frac{\mu_1\mathbf{i} - \mathrm{i}\mu_2\mathbf{j}}{(\mu_1^2-\mu_2^2)^{1/2}}, \qquad \mathbf{A}^- = \mathbf{k}, \qquad \delta^+ = \mathbf{A}^+\cdot\boldsymbol{\chi}\mathbf{A}^+ = \mathbf{A}^+\cdot\mathbb{B}^{-1}\mathbf{A}^+ = \frac{\mu_1^2\lambda_1^{-2}-\mu_2^2\lambda_2^{-2}}{\mu_1^2-\mu_2^2},$$
$$\delta^- = \mathbf{A}^-\cdot\boldsymbol{\chi}\mathbf{A}^- = \mathbf{A}^-\cdot\mathbb{B}^{-1}\mathbf{A}^- = \lambda_3^{-2}. \tag{4.15}$$

Using (4.9), we have

$$\mathbf{C}\cdot\mathbf{C} = \mu_2^2 - \mu_1^2 = a(\lambda_2^2-\lambda_1^2)(a+g'\lambda_3^2),$$
$$\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} = \mu_2^2\lambda_1^{-2} - \mu_1^2\lambda_2^{-2} = \frac{(\lambda_2^2-\lambda_1^2)[(g')^2 III + ag' II + a^2(\lambda_1^2+\lambda_2^2)]}{\lambda_1^2\lambda_2^2},$$
$$\rho N_{\parallel}^{-2} = (\lambda_2^2-\lambda_1^2) III (g' + a\lambda_3^{-2})(a^2 - bg'). \tag{4.16}$$

It then follows that the difference between the eigenvalues corresponding to $\mathbf{C}$ and $\mathbf{A}^-$ is

$$\rho N_{\parallel}^{-2} - b\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} - aIII\lambda_3^{-2}\mathbf{C}\cdot\mathbf{C} = (IIIg' + aII)\mathbf{C}\cdot\mathbf{C} - aIII\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} - aIII\lambda_3^{-2}\mathbf{C}\cdot\mathbf{C} = 0. \tag{4.17}$$

Thus the isotropic eigenbivector corresponding to the eigenvalue $\rho N_{\|}^{-2}$ for the above choice (4.13) of $\mathbf{C}$ is

$$\mathbf{A} = \frac{\mu_2\mathbf{i} + \mathrm{i}\mu_1\mathbf{j}}{(\mu_1^2 - \mu_2^2)^{1/2}} \pm \mathrm{i}\mathbf{k}. \tag{4.18}$$

Then, assuming that $a^2 - bg' > 0$, the corresponding explicit solution is given by

$$\begin{aligned}
\bar{x} &= x + \epsilon \left[\frac{\mu_2}{(\mu_2^2 - \mu_1^2)^{1/2}}\right] \mathrm{e}^{-\omega N_{\|}\mu_1 y} \cos\omega(N_{\|}\mu_2 x - t),\\
\bar{y} &= y + \epsilon \left[\frac{\mu_1}{(\mu_2^2 - \mu_1^2)^{1/2}}\right] \mathrm{e}^{-\omega N_{\|}\mu_1 y} \cos\omega(N_{\|}\mu_2 x - t),\\
\bar{z} &= z \pm \epsilon\, \mathrm{e}^{-\omega N_{\|}\mu_1 y} \sin\omega(N_{\|}\mu_2 x - t).
\end{aligned} \tag{4.19}$$

Here $(x, y, z) = (\lambda_1 X, \lambda_2 Y, \lambda_3 Z)$, $N_{\|}$ is given by (4.16) $_3$, and μ_1, μ_2 by (4.9). One radius of the circle of polarization is along $\mathbf{i}$, an orthogonal radius along $\mu_1\mathbf{j} \pm (\mu_1^2 - \mu_2^2)^{1/2}\mathbf{k}$. The waves travel in the x-direction with speed $(N_{\|}\mu_2)^{-1}$, and are attenuated in the y-direction.

4.3. *Remark: special case* $\mathbf{C} \cdot \mathbf{j} = 0$

It is shown (Appendix A, Case (i)) that $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are parallel if $\mathbf{C} \cdot \mathbf{j} = 0$. The eigenbivectors of $\boldsymbol{\chi}$ are then $\mathbf{C}^*$, $\mathbf{K}^+$, $\mathbf{j}$. Using (4.1), the eigenbivectors of $\mathbb{Q}(\mathbf{C})$ are also $\mathbf{C}^*$, $\mathbf{K}^+$, $\mathbf{j}$ in this case. However, because $C_2^* = 0$, $(C_1^*)^2 + (C_3^*)^2 = 1$, it follows that equating any two of the eigenvalues of $\mathbb{Q}(\mathbf{C})$ will lead to a $\mathbf{C}^*$ which is a 'real' bivector—a scalar multiple of a real vector.

The conclusion is therefore that if $\mathbf{C} \cdot \mathbf{j} = 0$, then the only possible circularly polarized waves are homogeneous.

4.4. *Case (iia): Transverse circularly polarized waves*

Here, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- \neq 0$ and $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, so that $\mathbf{\Pi n}^+$ is isotropic. The projection, of the ellipse of $\mathbf{C}$ upon the plane of central circular section of the $\mathbb{B}^{-1}$-ellipsoid with normal $\mathbf{n}^+$, is a circle.

Using the results in Appendix A relating to Case (iia), we conclude that $\mathbf{\Pi n}^+$ is an eigenbivector of $\mathbb{Q}(\mathbf{C})$ with eigenvalue $\rho N_{\perp}^{-2}$, given by

$$\rho N_{\perp}^{-2} = b(\mathbf{C} \cdot \mathbb{B}\mathbf{C}) + aIII(\mathbf{C} \cdot \mathbf{C})\nu_{\perp}, \quad \nu_{\perp} = \lambda_2^{-2} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})\mathbf{n}^+ \cdot \mathbf{\Pi n}^- = \tfrac{1}{2} I_{\boldsymbol{\chi}}. \tag{4.20}$$

See (A.15). The eigenvalue $\rho N_{\perp}^{-2}$ is a double root of the secular equation because one root is zero and the sum of the other roots is $2b(\mathbf{C} \cdot \mathbb{B}\mathbf{C}) + aIII(\mathbf{C} \cdot \mathbf{C}) I_{\boldsymbol{\chi}} = 2\rho N_{\perp}^{-2}$. All isotropic eigenbivectors, $\mathbf{A}_{\perp}$ (say), of $\mathbb{Q}(\mathbf{C})$, corresponding to $\rho N_{\perp}^{-2}$, are parallel to $\mathbf{\Pi n}^+$, and orthogonal to $\mathbf{C}$ (see Appendix A). Because $\mathbf{C} \cdot \mathbf{A}_{\perp} = 0$, we refer to these waves as transverse.

Using the fact that $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, we deduce that $\mathbf{C}$ has the form

$$\mathbf{C} = p\mathbf{n}^+ + s(\mathbf{j} \wedge \mathbf{n}^+ \pm \mathrm{i}\mathbf{j}), \tag{4.21}$$

where p and s are arbitrary constants. To relate this to the expression (3.3) for $\mathbf{C} = m\hat{\mathbf{m}} + \mathrm{i}\hat{\mathbf{n}}$, for which $\mathbf{C} \cdot \mathbf{C} = m^2 - 1$, $\mathbf{C} \cdot \bar{\mathbf{C}} = m^2 + 1$, we conclude that

$$\mathbf{C} = \sqrt{m^2 - 1}\mathbf{n}^+ + \mathrm{e}^{\mathrm{i}\theta}(\mathbf{j} \wedge \mathbf{n}^+ \pm \mathrm{i}\mathbf{j}) = (\alpha\sqrt{m^2 - 1} + \gamma\, \mathrm{e}^{\mathrm{i}\theta})\mathbf{i} \pm \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta}\mathbf{j} + (\gamma\sqrt{m^2 - 1} - \alpha\, \mathrm{e}^{\mathrm{i}\theta})\mathbf{k}, \tag{4.22}$$

where θ and m are arbitrary.

Also, because $\mathbf{A}_\perp$ is parallel to $\mathbf{\Pi n}^+$, it may be written

$$\mathbf{A}_\perp = \mathrm{e}^{\mathrm{i}\theta}(\mathbf{j} \wedge \mathbf{n}^+ + \mathrm{i}\mathbf{j}). \tag{4.23}$$

Now, using (2.14), we find that

$$\mathbf{C} \cdot \mathbb{B}\mathbf{C} = III\Lambda_\perp^2, \qquad \nu_\perp = \sqrt{m^2-1}\Lambda_\perp, \tag{4.24}$$

where

$$\Lambda_\perp = \sqrt{m^2-1}\lambda_2^{-2} + \mathrm{e}^{\mathrm{i}\theta}\sqrt{\lambda_2^{-2}-\lambda_1^{-2}}\sqrt{\lambda_3^{-2}-\lambda_2^{-2}}. \tag{4.25}$$

Hence, from (4.20),

$$\rho N_\perp^{-2} = III\Lambda_\perp(b\Lambda_\perp + a\sqrt{m^2-1}). \tag{4.26}$$

Remark. Forbidden choices of $\mathbf{C}$.

Throughout, it has been assumed that $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$ and $\mathbf{n}^- \cdot \mathbf{\Pi n}^- \neq 0$, and also $\mathbf{C} \cdot \mathbf{C} \neq 0$. The condition $\mathbf{C} \cdot \mathbf{C} \neq 0$ means that $\mathbf{C}$ given by (4.22) must be such that $m \neq 1$ or equivalently $\mathbf{C} \cdot \mathbf{n}^+ \neq 0$. The propagation bivector $\mathbf{C}$ given by (4.22) always satisfies $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$. However, as shown in Appendix A, Case (iii), $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$ also, if either $\mathbf{C} \cdot \mathbf{i} = 0$ or $\mathbf{C} \cdot \mathbf{k} = 0$. There are thus three forbidden choices of $\mathbf{C}$ given by (4.22): $\mathbf{C} \cdot \mathbf{i} = 0$ or $\mathbf{C} \cdot \mathbf{k} = 0$ or $\mathbf{C} \cdot \mathbf{n}^+ = 0$.

Any other choices of $\mathbf{C}$ given by (4.22) lead to circularly polarized transverse waves, all of which have the common amplitude bivector $\mathbf{A}_\perp$ given by (4.23). We note that $\mathbf{A}_\perp$ is determined by $\mathbf{n}^+$, which, in turn, is determined by the basic static homogeneous deformation. There is thus an infinity of circularly polarized transverse waves all propagating with common amplitude bivector $\mathbf{A}_\perp$ given by (4.23). We now present an example.

4.5. *Example: transverse circularly polarized waves*

We take $\theta = 0$, and then from Eqs. (4.22)–(4.25), we deduce the following circularly polarized plane wave solution

$$\bar{x} = x + \epsilon\sqrt{\frac{\lambda_2^{-2}-\lambda_1^{-2}}{\lambda_3^{-2}-\lambda_1^{-2}}}\,\mathrm{e}^{-\omega N_\perp y}\cos\omega(mN_\perp \mathbf{p}\cdot\mathbf{x} - t), \qquad \bar{y} = y - \epsilon\,\mathrm{e}^{-\omega N_\perp y}\sin\omega(mN_\perp \mathbf{p}\cdot\mathbf{x} - t),$$

$$\bar{z} = z - \epsilon\sqrt{\frac{\lambda_3^{-2}-\lambda_2^{-2}}{\lambda_3^{-2}-\lambda_1^{-2}}}\,\mathrm{e}^{-\omega N_\perp y}\cos\omega(mN_\perp \mathbf{p}\cdot\mathbf{x} - t). \tag{4.27}$$

Here (x, y, z) are the coordinates in the state of finite static deformation given by (2.10), $m \geq 1$ is arbitrary, $N_\perp$ is given by (4.22) $_3$, with $\Lambda_\perp = \lambda_2^{-2}\sqrt{m^2-1} + \sqrt{\lambda_2^{-2}-\lambda_1^{-2}}\sqrt{\lambda_3^{-2}-\lambda_2^{-2}}$, and $\mathbf{p}$ is the unit vector defined by

$$m\mathbf{p} = \mathbf{j} \wedge \mathbf{n}^+ + \sqrt{m^2-1}\mathbf{n}^+. \tag{4.28}$$

The radius of the circle of polarization at (x, y, z) is $\epsilon\,\mathrm{e}^{-\omega N_\perp y}$, two orthogonal radii being along $\mathbf{j}$ and $\mathbf{j} \wedge \mathbf{n}^+$. This wave propagates with speed $(mN_\perp)^{-1}$ in the direction of $\mathbf{p}$, and is attenuated in the direction of $\mathbf{j}$, orthogonal to $\mathbf{p}$. Note that *homogeneous* circularly polarized plane waves can only travel in the direction $\mathbf{n}^\pm$ of an acoustic axis [5]. In the present example, the normal to the planes of constant phase is $\mathbf{p}$, which may lie in any direction in the *xz*-plane, as m varies from 1 to ∞.

4.6. Case (iib): further transverse circularly polarized waves

Here, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$, $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ \neq 0$. This is identical to Case (iia), when $\mathbf{n}^+$ and $\mathbf{n}^-$ are interchanged, i.e. when γ is replaced by $-\gamma$. Accordingly, many details will be omitted.

Using the results in Appendix A relating to Case (iib), we conclude that the isotropic bivector $\mathbf{\Pi n}^-$ is an eigenbivector of $\mathbb{Q}(\mathbf{C})$ with eigenvalue $\rho \hat{N}_\perp^{-2}$ say, given by

$$\rho \hat{N}_\perp^{-2} = b(\mathbf{C} \cdot \mathbb{B}\mathbf{C}) + aIII(\mathbf{C} \cdot \mathbf{C})\hat{\nu}_\perp, \quad \hat{\nu}_\perp = \lambda_2^{-2} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})\mathbf{n}^+ \cdot \mathbf{\Pi n}^-, \tag{4.29}$$

precisely the same form as (4.20), but different in substance. All isotropic eigenbivectors $\hat{\mathbf{A}}_\perp$ (say) of $\mathbb{Q}(\mathbf{C})$, corresponding to $\rho \hat{N}_\perp^{-2}$, are parallel to $\mathbf{\Pi n}^-$, and orthogonal to $\mathbf{C}$.

As in Case (iia), we deduce the form of $\mathbf{C}$

$$\mathbf{C} = \hat{p}\mathbf{n}^- + \hat{s}(\mathbf{j} \wedge \mathbf{n}^- \pm \mathrm{i}\mathbf{j}) = (\alpha\sqrt{m^2 - 1} - \gamma\, \mathrm{e}^{\mathrm{i}\theta})\mathbf{i} \pm \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta}\mathbf{j} - (\gamma\sqrt{m^2 - 1} + \alpha\, \mathrm{e}^{\mathrm{i}\theta})\mathbf{k}. \tag{4.30}$$

Also,

$$\hat{\mathbf{A}}_\perp = \mathrm{e}^{\mathrm{i}\theta}(\mathbf{j} \wedge \mathbf{n}^- + \mathrm{i}\mathbf{j}), \tag{4.31}$$

and

$$\mathbf{C} \cdot \mathbb{B}\mathbf{C} = III\hat{\Lambda}_\perp^2, \qquad \hat{\nu}_\perp = \sqrt{m^2 - 1}\,\hat{\Lambda}_\perp, \tag{4.32}$$

where

$$\hat{\Lambda}_\perp = \sqrt{m^2 - 1}\,\lambda_2^{-2} - \mathrm{e}^{\mathrm{i}\theta}\sqrt{\lambda_2^{-2} - \lambda_1^{-2}}\sqrt{\lambda_3^{-2} - \lambda_2^{-2}}. \tag{4.33}$$

Then

$$\rho \hat{N}_\perp^{-2} = III\hat{\Lambda}_\perp(b\hat{\Lambda}_\perp + a\sqrt{m^2 - 1}). \tag{4.34}$$

Thus, as in Case (iib), apart from the three cases when $\mathbf{C} \cdot \mathbf{i} = 0$ or $\mathbf{C} \cdot \mathbf{k} = 0$ or $\mathbf{C} \cdot \mathbf{n}^- = 0$, any choice of $\mathbf{C}$ given by (4.30) will lead to a circularly polarized transverse wave. There is an infinity of such waves, all sharing a common amplitude bivector $\hat{\mathbf{A}}_\perp$ given by (4.31), which is determined by the basic static homogeneous deformation.

4.7. Case (iii): principal circularly polarized waves

Here $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$, so that both $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are isotropic. As shown in Appendix A, Case (iii), the propagation bivector $\mathbf{C}$ must satisfy either

$$\text{(a)} \quad \mathbf{C} \cdot \mathbf{i} = 0 \quad \text{or} \quad \text{(b)} \quad \mathbf{C} \cdot \mathbf{k} = 0. \tag{4.35}$$

It is not possible to have both $\mathbf{C} \cdot \mathbf{i} = 0$, and $\mathbf{C} \cdot \mathbf{k} = 0$, because then $\mathbf{C} \wedge \mathbf{j} = \mathbf{0}$, and $(\mathbf{C} \cdot \mathbf{n}^\pm)^2 = 0$, $\mathbf{C} \cdot \mathbf{C} \neq 0$. The possible forms for $\mathbf{C}$ are

$$\mathbf{C} = \mathbf{k} \pm \mathrm{i}\alpha\mathbf{j}, \tag{4.36a}$$

and

$$\mathbf{C} = \mathbf{i} \pm \mathrm{i}\gamma\mathbf{j}. \tag{4.36b}$$

As shown in Appendix A, $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are linearly independent eigenbivectors of $\boldsymbol{\chi}$, with common eigenvalue λ_2^{-2}. So these bivectors $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are isotropic eigenbivectors of the acoustical tensor $\mathbb{Q}(\mathbf{C})$, with eigenvalues ρN_a^{-2}, ρN_b^{-2} (say) given by

$$\rho N_a^{-2} = b(\mathbf{C} \cdot \mathbb{B}\mathbf{C}) + aIII(\mathbf{C} \cdot \mathbf{C})\lambda_1^{-2} = \frac{(b + a\lambda_1^2)\lambda_3^2(\lambda_2^2 - \lambda_3^2)}{\lambda_1^2 - \lambda_3^2}, \tag{4.37a}$$

and

$$\rho N_b^{-2} = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) + aIII(\mathbf{C}\cdot\mathbf{C})\lambda_3^{-2} = \frac{(b+a\lambda_3^2)\lambda_1^2(\lambda_1^2-\lambda_2^2)}{\lambda_1^2-\lambda_3^2}. \tag{4.37b}$$

Thus, in Case (a), corresponding to $\mathbf{C}$ given by (4.36a), there are three eigenbivectors for $\mathbb{Q}(\mathbf{C})$, namely $\mathbf{C}$ given by (4.36a) with eigenvalue zero, $\mathbf{\Pi n}^+$ with eigenvalue ρN_a^{-2}, and $\mathbf{\Pi n}^-$, also with eigenvalue ρN_a^{-2}. Here, $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are, respectively, parallel to the bivectors $\mathbf{A}_1$ and $\mathbf{A}_2$ say, given by

$$\mathbf{A}_1 = \gamma\mathbf{i} - \alpha\mathbf{k} \mp \mathrm{i}\mathbf{j}, \qquad \mathbf{A}_2 = \gamma\mathbf{i} + \alpha\mathbf{k} \pm \mathrm{i}\mathbf{j}. \tag{4.38}$$

Similarly, in Case (b), there are three eigenbivectors for $\mathbb{Q}(\mathbf{C}) - \mathbf{C}$ given by (4.36b), and $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$, also with common eigenvalue ρN_b^{-2}. Here, the bivectors $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are again parallel to the bivectors $\mathbf{A}_1$ and $\mathbf{A}_2$ given by (4.38).

It may be noted that there are two circularly polarized waves with amplitude bivector along $\mathbf{\Pi n}^+$ given by (4.38), one with propagation bivector given by (4.36a), and complex slowness given by (4.37a), the other with $\mathbf{C}$ given by (4.36b), and complex slowness given by (4.37b). Similarly, there are two circularly polarized waves with amplitude bivector along $\mathbf{\Pi n}^-$ given by (4.38).

Using (2.6), we note that ρN_a^{-2}, ρN_b^{-2} are both real and positive. Consequently, the direction of propagation, which in general is parallel to the real part of $N\mathbf{C}$, see (3.1), is here parallel to $\mathbf{C}^+$, i.e. $\mathbf{k}$ in Case (a) or $\mathbf{i}$ in Case (b). Similarly, the direction of attenuation, which in general is parallel to the imaginary part of $N\mathbf{C}$, see (3.1), is here parallel to $\mathbf{C}^-$, i.e. $\mathbf{j}$. Hence, the direction of propagation is either along the principal axis corresponding to the largest strain or along the principal axis corresponding to the least strain, whilst the direction of attenuation is along the intermediate axis. Such inhomogeneous waves for which the planes of constant phase and the planes of constant amplitude are normal to principal axes of the basic homogeneous strain, may be called 'principal' waves [14]. Here, their circle of polarization has a radius along the intermediate axis and orthogonal radii along either $\mathbf{n}^+ \wedge \mathbf{j}$ or $\mathbf{n}^- \wedge \mathbf{j}$. Here is a specific example.

Example (Principal circularly polarized waves). Let $\mathbf{C} = \mathbf{i} + \mathrm{i}\gamma\mathbf{j}$.

In this case, we find the following two special principal circularly polarized waves,

$$\begin{aligned} &\bar{x} = x + \epsilon\gamma\, \mathrm{e}^{-\omega N_b \gamma y}\cos\omega(N_b x - t), \qquad \bar{y} = y \pm \epsilon\, \mathrm{e}^{-\omega N_b \gamma y}\sin\omega(N_b x - t),\\ &\bar{z} = z \mp \epsilon\alpha\, \mathrm{e}^{-\omega N_b \gamma y}\cos\omega(N_b x - t), \end{aligned} \tag{4.39}$$

where $(x, y, z) = (\lambda_1 X, \lambda_2 Y, \lambda_3 Z)$ and ω is arbitrary. For these waves, the propagation is in the direction of $\mathbf{i}$, the attenuation in the direction of $\mathbf{j}$, and the speed is N_b^{-1}, where N_b is given by (4.37b).

Hence, we have investigated all possible circularly polarized inhomogeneous plane waves of small-amplitude propagating in a finitely deformed Hadamard material, with a non-isotropic bivector $\mathbf{C}$. There are two types of such solutions. One type corresponds to waves whose amplitude bivector $\mathbf{A}$ is orthogonal to the propagation bivector $\mathbf{C}$; another type corresponds to waves whose amplitude bivector $\mathbf{A}$ is the sum of a bivector orthogonal to $\mathbf{C}$ and a bivector parallel to $\mathbf{C}$.

Central to this investigation was the use of the tensor $\mathbf{\Pi}$ given by (4.2) for which it was assumed that $\mathbf{C}\cdot\mathbf{C} \neq 0$. Now we examine in detail the cases where $\mathbf{C}\cdot\mathbf{C} = 0$.

5. Circularly polarized inhomogeneous plane waves with an isotropic propagation bivector C

Here we seek circularly polarized inhomogeneous plane waves with an isotropic bivector $\mathbf{C}$. It is seen that such waves can propagate in the deformed Hadamard material as long as the circle of $\mathbf{C}$ is not similarly situated to either of the central circular sections of the $\mathbb{B}^{-1}$-ellipsoid.

5.1. The acoustical tensor

When $\mathbf{C}\cdot\mathbf{C}=0$, the acoustical tensor $\mathbb{Q}(\mathbf{C})$ given by (3.15) reduces to

$$\mathbb{Q}(\mathbf{C}) = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C})\mathbf{1} + (IIIg' + aII)\mathbf{C}\otimes\mathbf{C} - aIII[\mathbf{C}\otimes\mathbb{B}^{-1}\mathbf{C} + \mathbb{B}^{-1}\mathbf{C}\otimes\mathbf{C}], \tag{5.1}$$

and $\mathbf{C}$ is an isotropic eigenbivector of $\mathbb{Q}(\mathbf{C})$ with eigenvalue ρN_0^{-2} (say), given by

$$\rho N_0^{-2} = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) - aIII(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}). \tag{5.2}$$

Because the eigenbivector $\mathbf{C}$ is isotropic, the eigenvalue ρN_0^{-2} is at least double [11]. Let ρN_1^{-2} be the remaining eigenvalue of $\mathbb{Q}(\mathbf{C})$, possibly equal to ρN_0^{-2}. This quantity can be deduced from the equality $\operatorname{tr}\mathbb{Q}(\mathbf{C}) = 2\rho N_0^{-2} + \rho N_1^{-2}$, using (5.1) and (5.2). It is given by

$$\rho N_1^{-2} = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C}). \tag{5.3}$$

Also, the bivector $\mathbf{C}\wedge\mathbb{B}^{-1}\mathbf{C}$ is clearly an eigenbivector of $\mathbb{Q}(\mathbf{C})$, with eigenvalue ρN_1^{-2}.

Hence, we see that the two eigenvalues given by (5.2) and (5.3) are distinct or equal according as to whether or not $\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}$ is equal to zero. Consequently, we consider in turn the following cases:

1. $\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}=0$, $\mathbf{C}\cdot\mathbf{C}=0$;
2. $\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}\neq 0$, $\mathbf{C}\cdot\mathbf{C}=0$.

5.2. Case (1): $\mathbf{C}\cdot\mathbf{C}=\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}=0$*: longitudinal circularly polarized waves*

In this case, $\rho N_0^{-2} = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C})$ is a triple eigenvalue of $\mathbb{Q}(\mathbf{C})$. The condition

$$\mathbf{C}\cdot\mathbf{C}=\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}=0, \tag{5.4}$$

means that the plane of $\mathbf{C}$ must coincide with either plane of central circular section of the $\mathbb{B}^{-1}$-ellipsoid [11]. Let $\mathbf{A}$ be an eigenbivector of $\mathbb{Q}(\mathbf{C})$. Then $\mathbb{Q}(\mathbf{C})\mathbf{A} = \rho N_0^{-2}\mathbf{A}$ is equivalent to

$$[aIII(\mathbf{A}\cdot\mathbf{C})]\mathbb{B}^{-1}\mathbf{C} = [(IIIg' + aII)(\mathbf{A}\cdot\mathbf{C}) - aIII(\mathbf{A}\cdot\mathbb{B}^{-1}\mathbf{C})]\mathbf{C}. \tag{5.5}$$

However, $\mathbb{B}^{-1}\mathbf{C}$ and $\mathbf{C}$ can never be parallel. Indeed, if we had $\mathbb{B}^{-1}\mathbf{C} = \lambda^{-2}\mathbf{C}$ for $\mathbf{C} = \hat{\mathbf{m}} + \mathrm{i}\hat{\mathbf{n}}$, then $\mathbb{B}^{-1}\hat{\mathbf{m}} = \lambda^{-2}\hat{\mathbf{m}}$, $\mathbb{B}^{-1}\hat{\mathbf{n}} = \lambda^{-2}\hat{\mathbf{n}}$, and λ^{-2} would have to be a double real eigenvalue for $\mathbb{B}^{-1}$, which is not possible. So, the coefficients of $\mathbb{B}^{-1}\mathbf{C}$ and $\mathbf{C}$ in (5.5) must be zero, which yield in turn

$$\mathbf{A}\cdot\mathbf{C} = 0 \quad \text{and} \quad \mathbf{A}\cdot\mathbb{B}^{-1}\mathbf{C} = 0. \tag{5.6}$$

Therefore, $\mathbf{A}$ is parallel to $\mathbb{B}^{-1}\mathbf{C}\wedge\mathbf{C}$, which itself is parallel to $\mathbf{C}$.

We conclude that when (5.4) holds, all eigenbivectors of $\mathbb{Q}(\mathbf{C})$ are isotropic and parallel to $\mathbf{C}$. The corresponding circularly polarized waves are said to be longitudinal.

Example (Longitudinal circularly polarized waves). We let $\mathbf{C} = \mathbf{p} + \mathrm{i}\mathbf{j}$, where $\mathbf{p} = \gamma\mathbf{i} + \alpha\mathbf{k}$.

With this choice, (5.4) is satisfied, and all amplitude bivectors are parallel to $\mathbf{C}$. The corresponding eigenvalue ρN_0^{-2} is given by

$$\rho N_0^{-2} = b(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) = b\lambda_2^{-2}(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2) > 0. \tag{5.7}$$

Thus an example of a longitudinal circularly polarized wave propagating in a deformed Hadamard material is

$$\bar{x} = x + \epsilon\gamma\,\mathrm{e}^{-\omega N_0 y}\cos\omega(N_0\mathbf{p}\cdot\mathbf{x} - t), \qquad \bar{y} = y - \epsilon\,\mathrm{e}^{-\omega N_0 y}\sin\omega(N_0\mathbf{p}\cdot\mathbf{x} - t),$$
$$\bar{z} = z + \epsilon\alpha\,\mathrm{e}^{-\omega N_0 y}\cos\omega(N_0\mathbf{p}\cdot\mathbf{x} - t). \tag{5.8}$$

Here $(x, y, z) = (\lambda_1 X, \lambda_2 Y, \lambda_3 Z)$ and ω is arbitrary. This wave travels in the direction of $\mathbf{p}$ with speed N_0^{-1} given by (5.7), and is attenuated in the direction of $\mathbf{j}$. The vectors $\mathbf{p}$ and $\mathbf{j}$ are two orthogonal radii of the circle of polarization.

5.3. *Case (2):* $\mathbf{C}\cdot\mathbf{C} = 0$, $\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} \neq 0$. *Other circularly polarized waves*

Because the ellipse of $\mathbf{C}$ is a circle, the condition

$$\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} \neq 0, \tag{5.9}$$

means that $\mathbf{C}$ may not lie in either plane of central circular sections of the $\mathbb{B}^{-1}$-ellipsoid. An immediate consequence of this condition and of the isotropy of $\mathbf{C}$ is that the bivectors $\mathbf{C}$, $\mathbb{B}^{-1}\mathbf{C}$, and $\mathbf{C}\wedge\mathbb{B}^{-1}\mathbf{C}$ are linearly independent, so that any bivector may be written as a linear combination of $\mathbf{C}$, $\mathbb{B}^{-1}\mathbf{C}$, and $\mathbf{C}\wedge\mathbb{B}^{-1}\mathbf{C}$. In particular, the eigenbivectors $\mathbf{A}$ of $\mathbb{Q}(\mathbf{C})$ with eigenvalue ρN_0^{-2} must be orthogonal to $\mathbf{C}\wedge\mathbb{B}^{-1}\mathbf{C}$, the eigenbivector of $\mathbb{Q}(\mathbf{C})$ with eigenvalue ρN_1^{-2} $(\neq \rho N_0^{-2})$, and may therefore be written as

$$\mathbf{A} = \alpha_1\mathbf{C} + \alpha_2\mathbb{B}^{-1}\mathbf{C} \tag{5.10}$$

for some complex scalars α_1 and α_2. We seek to determine α_1 and α_2.

Now, $\mathbb{Q}(\mathbf{C})\mathbf{A} = \rho N_0^{-2}\mathbf{A}$ yields

$$\{(g')III(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}) + a[II(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}) - III(\mathbf{C}\cdot\mathbb{B}^{-2}\mathbf{C})]\}\alpha_2 = 0 \tag{5.11}$$

or using the Cayley–Hamilton theorem and the isotropy of $\mathbf{C}$,

$$(\mathbf{C}\cdot\mathbf{\Phi}\mathbf{C})\alpha_2 = 0, \quad \text{where} \quad \mathbf{\Phi} = (g')III\mathbb{B}^{-1} - a\mathbb{B}. \tag{5.12}$$

Therefore, there are two possibilities for the eigenbivectors of $\mathbb{Q}(\mathbf{C})$:

Case (a) $\alpha_2 = 0$, and the eigenbivectors $\mathbf{A} = \alpha_1\mathbf{C}$ are all isotropic and parallel to $\mathbf{C}$, and

Case (b) $\alpha_2 \neq 0$, $\mathbf{C}\cdot\mathbf{\Phi}\mathbf{C} = 0$, and there is a double infinity of eigenbivectors, of the form (5.10).

Now we examine the consequences of this result for the possibility of circular polarization.

In Case (a), all eigenbivectors of the acoustical tensor with the eigenvalue ρN_0^{-2} are isotropic and the corresponding waves are therefore circularly polarized. They are longitudinal waves, in the sense that the amplitude bivector $\mathbf{A}$ is parallel to the propagation bivector $\mathbf{C}$.

In Case (b), α_2 is arbitrary and we choose it to make $\mathbf{A}$ given by (5.10) isotropic. We have, up to a complex factor,

$$\mathbf{A} = \mathbf{C} \quad \text{and} \quad \mathbf{A} = (\mathbf{C}\cdot\mathbb{B}^{-2}\mathbf{C})\mathbf{C} - 2(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C})\mathbb{B}^{-1}\mathbf{C}. \tag{5.13}$$

Also, we note that it is always possible to find an isotropic bivector $\mathbf{C}$ such that the equation $\mathbf{C}\cdot\mathbf{\Phi}\mathbf{C} = 0$ is satisfied. Indeed, because $\mathbf{C}\cdot\mathbf{C} = 0$, this equation can be written as $\mathbf{C}\cdot[\mathbf{\Phi} + \beta^2\mathbf{1}]\mathbf{C} = 0$, where β is an arbitrary real scalar. By choosing β sufficiently large, we can ensure that the diagonal tensor $[\mathbf{\Phi} + \beta^2\mathbf{1}]$ is positive definite, and prescribe the circle of $\mathbf{C}$ to lie in either of the planes of central circular sections of the ellipsoid $\mathbf{x}\cdot[\mathbf{\Phi} + \beta^2\mathbf{1}]\mathbf{x} = 1$.

As an example, we prescribe an isotropic bivector $\mathbf{C}$ and write the corresponding longitudinal circularly polarized wave.

Example (Longitudinal circularly polarized wave). Let $\mathbf{C} = \mathbf{q} + \mathrm{i}\mathbf{j}$, where the real unit vector $\mathbf{q}$ is defined by

$$\mathbf{q} = \sqrt{\frac{(\lambda_2^2 - \lambda_3^2)(g'\lambda_1^2 + a)}{(\lambda_1^2 - \lambda_3^2)(g'\lambda_2^2 + a)}}\mathbf{i} + \sqrt{\frac{(\lambda_1^2 - \lambda_2^2)(g'\lambda_3^2 + a)}{(\lambda_1^2 - \lambda_3^2)(g'\lambda_2^2 + a)}}\mathbf{k}.$$

It can be checked that this bivector $\mathbf{C}$ satisfies $\mathbf{C} \cdot \mathbb{B}^{-1}\mathbf{C} \neq 0$ and $\mathbf{C} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{\Phi}\mathbf{C} = 0$.

The eigenvalue ρN_0^{-2} given by (5.2) reduces to

$$\rho N_0^{-2} = \frac{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(bg' - a^2)}{g'\lambda_2^2 + a}, \tag{5.14}$$

and turns out to be real. Of course, its sign depends on whether bg' is greater or smaller than a^2. We consider these two possibilities in turn and introduce the quantity v_0, which has the dimension of a speed, and is defined by

$$\rho v_0^2 = \frac{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)|bg' - a^2|}{g'\lambda_2^2 + a}. \tag{5.15}$$

If $bg' > a^2$, then a solution to the incremental equations of motion in a deformed Hadamard material, corresponding to a longitudinal circularly polarized wave, is given by

$$\bar{x} = x + \epsilon\sqrt{\frac{(\lambda_2^2 - \lambda_3^2)(g'\lambda_1^2 + a)}{(\lambda_1^2 - \lambda_3^2)(g'\lambda_2^2 + a)}}\,\mathrm{e}^{-ky}\cos k(\mathbf{q}\cdot\mathbf{x} - v_0 t), \qquad \bar{y} = y - \epsilon\,\mathrm{e}^{-ky}\sin k(\mathbf{q}\cdot\mathbf{x} - v_0 t),$$
$$\bar{z} = z + \epsilon\sqrt{\frac{(\lambda_1^2 - \lambda_2^2)(g'\lambda_3^2 + a)}{(\lambda_1^2 - \lambda_3^2)(g'\lambda_2^2 + a)}}\,\mathrm{e}^{-ky}\cos k(\mathbf{q}\cdot\mathbf{x} - v_0 t). \tag{5.16}$$

If $bg' < a^2$, then a solution is given by

$$\bar{x} = x + \epsilon\sqrt{\frac{(\lambda_2^2 - \lambda_3^2)(g'\lambda_1^2 + a)}{(\lambda_1^2 - \lambda_3^2)(g'\lambda_2^2 + a)}}\,\mathrm{e}^{-k\mathbf{q}\cdot\mathbf{x}}\cos k(y - v_0 t), \qquad \bar{y} = y - \epsilon\,\mathrm{e}^{-k\mathbf{q}\cdot\mathbf{x}}\sin k(y - v_0 t),$$
$$\bar{z} = z + \epsilon\sqrt{\frac{(\lambda_1^2 - \lambda_2^2)(g'\lambda_3^2 + a)}{(\lambda_1^2 - \lambda_3^2)(g'\lambda_2^2 + a)}}\,\mathrm{e}^{-k\mathbf{q}\cdot\mathbf{x}}\cos k(y - v_0 t). \tag{5.17}$$

In both cases, $(x, y, z) = (\lambda_1 X, \lambda_2 Y, \lambda_3 Z)$ and k is arbitrary. When $bg' > a^2$, the wave propagates in the direction of $\mathbf{q}$ and is attenuated in the direction of $\mathbf{j}$; when $bg' < a^2$, the wave propagates in the direction of $\mathbf{j}$ and is attenuated in the direction of $\mathbf{q}$. In both cases, the wave travels with speed v_0 given by (5.15) and is circularly polarized, the vectors $\mathbf{q}$ and $\mathbf{j}$ being two orthogonal radii of the circle of polarization.

6. Concluding remarks: homogeneous plane waves

The analysis carried above (Sections 4 and 5) can be applied to the consideration of homogeneous plane waves, simply by taking

$$\mathbf{C} = \mathbf{n}, \tag{6.1}$$

where $\mathbf{n}$ is a real unit vector in the direction of propagation of the wave. However, the case of an isotropic propagation vector $\mathbf{C} \cdot \mathbf{C} = 0$ (Section 5) does not arise, because now $\mathbf{C} \cdot \mathbf{C} = \mathbf{n} \cdot \mathbf{n} = 1$. Also, transverse and longitudinal waves cannot be superposed (Section 4.1) to form a circularly polarized homogeneous wave because the condition (4.7), which reduces here to $\mathbf{C} \cdot \mathbb{M}\mathbf{C} = \mathbf{n} \cdot \mathbb{M}\mathbf{n} = n_1^2\mu_1^2 + n_2^2\mu_2^2 + n_3^2\mu_3^2 = 0$, cannot be satisfied. Now, for completeness, we consider briefly the propagation of circularly polarized homogeneous plane waves.

First, we recall that homogeneous longitudinal plane waves may propagate in every direction $\mathbf{n}$ in a Hadamard material maintained in a state of finite static homogeneous deformation. Two transverse plane waves may also propagate in every direction $\mathbf{n}$. If the corresponding directions of polarization are along $\mathbf{h}$, $\mathbf{l}$, forming an orthonormal triad with $\mathbf{n}$, then it has been shown [5] that $\mathbf{h} \cdot \mathbb{B}^{-1}\mathbf{l} = 0$, so that $\mathbf{h}$ and $\mathbf{l}$ must lie along the principal axes of the elliptical section of the $\mathbb{B}^{-1}$-ellipsoid $\mathbf{x} \cdot \mathbb{B}^{-1}\mathbf{x} = 1$ by the central plane $\mathbf{n} \cdot \mathbf{x} = 0$. Here we present a simple derivation of this result.

The acoustical tensor $\mathbb{Q}(\mathbf{n})$ for homogeneous plane waves may be obtained from the expression (3.15) for $\mathbb{Q}(\mathbf{C})$, by replacing $\mathbf{C}$ with $\mathbf{n}$. Indeed

$$\mathbb{Q}(\mathbf{n}) = b(\mathbf{n} \cdot \mathbb{B}\mathbf{n})\mathbf{1} + (IIIg' + aII)\mathbf{n} \otimes \mathbf{n} + aIII(\mathbb{B}^{-1} - \mathbf{n} \otimes \mathbb{B}^{-1}\mathbf{n} - \mathbb{B}^{-1}\mathbf{n} \otimes \mathbf{n}), \tag{6.2}$$

and

$$\mathbb{Q}(\mathbf{n})\mathbf{n} = [b(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) + IIIg' + aII - aIII(\mathbf{n} \cdot \mathbb{B}^{-1}\mathbf{n})]\mathbf{n}. \tag{6.3}$$

Because $\mathbf{h}$, $\mathbf{l}$ are eigenvectors of $\mathbb{Q}(\mathbf{n})$, both orthogonal to $\mathbf{n}$, we have

$$\mathbb{Q}(\mathbf{n})\mathbf{h} = [b(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) + aIII(\mathbf{h} \cdot \mathbb{B}^{-1}\mathbf{h})]\mathbf{h}, \tag{6.4}$$

$$\mathbb{Q}(\mathbf{n})\mathbf{l} = [b(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) + aIII(\mathbf{l} \cdot \mathbb{B}^{-1}\mathbf{l})]\mathbf{l}. \tag{6.5}$$

Thus, for circularly polarized homogeneous plane waves, $\mathbf{n}$ must be such that

$$\mathbf{h} \cdot \mathbb{B}^{-1}\mathbf{h} = \mathbf{l} \cdot \mathbb{B}^{-1}\mathbf{l}, \qquad \mathbf{h} \cdot \mathbb{B}^{-1}\mathbf{l} = 0, \qquad \mathbf{h} \cdot \mathbf{l} = 0, \tag{6.6}$$

and hence, $\mathbf{n} = \mathbf{n}^{\pm}$, the normals to the planes of central circular sections of the $\mathbb{B}^{-1}$-ellipsoid. The corresponding speed of propagation $v_\perp$ say, is given by

$$\rho v_\perp^2 = b(\mathbf{n} \cdot \mathbb{B}\mathbf{n}) + aIII\lambda_2^{-2} = \lambda_2^{-2}(b\lambda_2^{-2} + aIII). \tag{6.7}$$

This result was established for finite-amplitude plane waves by Boulanger et al. [5]. We note that on replacing $\mathbf{C}$ by $\mathbf{n}^{\pm}$ in (4.29), $\nu_\perp$ becomes λ_2^{-2}, and then we obtain (6.7) on replacing $N_\perp^{-2}$ by $v_\perp^2$.

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Appendix A. Properties of the complex projection operator

Here we present some properties of the complex projection operator Π, defined in (4.2). In particular, we determine the eigenvalues and eigenbivectors of $\boldsymbol{\chi} = \Pi\mathbb{B}^{-1}\Pi$, the tensor which arises in the expression (4.1) for the acoustical tensor $\mathbb{Q}(\mathbf{C})$. Many of these properties may be found in [13], but are included here for completeness. The material is self-contained.

If $\mathbf{h}$ is a real unit vector, then $\boldsymbol{\phi}$ defined by

$$\boldsymbol{\phi} = \mathbf{1} - \mathbf{h} \otimes \mathbf{h}, \qquad \phi_{ij} = \delta_{ij} - h_i h_j, \tag{A.1}$$

is a projection operator with the properties

$$\boldsymbol{\phi}^2 = \boldsymbol{\phi}, \quad \boldsymbol{\phi}\,\mathbf{h} = \mathbf{0}, \qquad \boldsymbol{\phi}\,\mathbf{l} = \mathbf{l} \quad \forall \mathbf{l} : \mathbf{l} \cdot \mathbf{h} = 0. \tag{A.2}$$

For any second order tensor $\mathbf{g}$, the projection of $\mathbf{g}$ on the plane with unit normal $\mathbf{h}$, is $\boldsymbol{\phi}\,\mathbf{g}\,\boldsymbol{\phi}$, with components

$$(\boldsymbol{\phi}\,\mathbf{g}\,\boldsymbol{\phi})_{ij} = \phi_{im} g_{mp} \phi_{pj} = (\delta_{im} - h_i h_m) g_{mp} (\delta_{pj} - h_p h_j). \tag{A.3}$$

In the case of bivectors, let $\mathbf{C}$ be a non-isotropic bivector: $\mathbf{C} \cdot \mathbf{C} \neq 0$. Let $\mathbf{C}^* = \mathbf{C}/(\mathbf{C} \cdot \mathbf{C})^{1/2}$. Then the complex projection operator $\boldsymbol{\Pi}$, defined by

$$\boldsymbol{\Pi} = \mathbf{1} - \mathbf{C}^* \otimes \mathbf{C}^*, \tag{A.4}$$

is such that corresponding to (A.1) for the real projection operator $\boldsymbol{\phi}$, we have

$$\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}, \quad \boldsymbol{\Pi}\mathbf{C}^* = \mathbf{0}, \qquad \boldsymbol{\Pi}\mathbf{D} = \mathbf{D} \quad \forall \mathbf{D} : \mathbf{D} \cdot \mathbf{C} = 0. \tag{A.5}$$

We consider the complex tensor $\boldsymbol{\chi}$, defined by

$$\boldsymbol{\chi} = \boldsymbol{\Pi}\mathbb{B}^{-1}\boldsymbol{\Pi}, \qquad \chi_{ij} = (\delta_{im} - C_i^* C_m^*)\mathbb{B}^{-1}_{mp}(\delta_{pj} - C_p^* C_j^*), \tag{A.6}$$

where $\mathbb{B}^{-1}$ is real, positive definite, and symmetric.

A.1. General properties of $\boldsymbol{\chi}$

Recalling Eq. (2.15)

$$\mathbb{B}^{-1} = \lambda_2^{-2}\mathbf{1} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})[\mathbf{n}^+ \otimes \mathbf{n}^- + \mathbf{n}^- \otimes \mathbf{n}^+], \tag{A.7}$$

it follows that

$$\boldsymbol{\chi} = \lambda_2^{-2}\boldsymbol{\Pi} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})[\boldsymbol{\Pi}\mathbf{n}^+ \otimes \boldsymbol{\Pi}\mathbf{n}^- + \boldsymbol{\Pi}\mathbf{n}^- \otimes \boldsymbol{\Pi}\mathbf{n}^+]. \tag{A.8}$$

Equivalently, provided $\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+ \neq 0$, $\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^- \neq 0$, $\boldsymbol{\chi}$ may be written

$$\boldsymbol{\chi} = \lambda_2^{-2}\boldsymbol{\Pi} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})[(\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+)(\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^-)]^{1/2}[\mathbf{K}^+ \otimes \mathbf{K}^- + \mathbf{K}^- \otimes \mathbf{K}^+], \tag{A.9}$$

where $\mathbf{K}^{\pm}$ are unit bivectors given by

$$\mathbf{K}^{\pm} = \frac{\boldsymbol{\Pi}\mathbf{n}^{\pm}}{(\mathbf{n}^{\pm} \cdot \boldsymbol{\Pi}\mathbf{n}^{\pm})^{1/2}}. \tag{A.10}$$

A.2. Eigenvalues of $\boldsymbol{\chi}$

We note that $\boldsymbol{\chi}$ is symmetric. Also $\boldsymbol{\chi}\mathbf{C}^* = \mathbf{0}$, so that $\boldsymbol{\chi}$ has one zero eigenvalue corresponding to its eigenbivector $\mathbf{C}^*$. The other eigenvalues are the roots, α, of the quadratic,

$$\alpha^2 - I_{\boldsymbol{\chi}}\alpha + II_{\boldsymbol{\chi}} = 0, \tag{A.11}$$

where

$$\begin{aligned} I_{\boldsymbol{\chi}} &= \operatorname{tr}\boldsymbol{\chi} = \operatorname{tr}(\boldsymbol{\Pi}\mathbb{B}^{-1}\boldsymbol{\Pi}) = \operatorname{tr}\mathbb{B}^{-1} - \mathbf{C}^* \cdot \mathbb{B}^{-1}\mathbf{C}^*, \\ 2II_{\boldsymbol{\chi}} &= (I_{\boldsymbol{\chi}})^2 - \operatorname{tr}(\boldsymbol{\chi}^2) = (I_{\boldsymbol{\chi}})^2 - \operatorname{tr}(\mathbb{B}^{-2}) + 2\mathbf{C}^* \cdot \mathbb{B}^{-1}\mathbf{C}^* - (\mathbf{C}^* \cdot \mathbb{B}^{-1}\mathbf{C}^*)^2 = \frac{2(\mathbf{C}^* \cdot \mathbb{B}^{-1}\mathbf{C}^*)}{III}. \end{aligned} \tag{A.12}$$

The condition that the quadratic have a double root for α is that

$$I_{\boldsymbol{\chi}}^2 = 4II_{\boldsymbol{\chi}}, \tag{A.13}$$

or equivalently,

$$III(\mathrm{tr}\,\mathbb{B}^{-1} - \mathbf{C}^* \cdot \mathbb{B}^{-1}\mathbf{C}^*)^2 = 4\mathbf{C}^* \cdot \mathbb{B}\mathbf{C}^*. \tag{A.14}$$

If this is satisfied, the double root of the quadratic is $\hat{\alpha}$ (say), given by

$$\hat{\alpha} = \frac{1}{2} I_{\boldsymbol{\chi}} = f\frac{1}{2}(\mathrm{tr}\,\mathbb{B}^{-1} - \mathbf{C}^* \cdot \mathbb{B}^{-1}\mathbf{C}^*) = \frac{2(\mathbf{C}^* \cdot \mathbb{B}^{-1}\mathbf{C}^*)}{III} = \lambda_2^{-2} - \frac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})(\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^-), \tag{A.15}$$

on using (A.8).

The quadratic has a zero root provided $II_{\boldsymbol{\chi}} = 0$ or

$$\mathbf{C}^* \cdot \mathbb{B}\mathbf{C}^* = 0, \tag{A.16}$$

in which case $\mathbf{C}^*$ is any bivector whose ellipse is similar and similarly situated to a section of the $\mathbb{B}$-ellipsoid, other than a central circular section (throughout this Appendix A, it is assumed that $\mathbf{C}$ is not an isotropic bivector). There is thus an infinity of possible $\mathbf{C}^*$ satisfying (A.16)—or equivalently an infinity of possible choices of $\mathbf{C}^*$ for which $\boldsymbol{\chi}$ has two zero eigenvalues. In this instance, the third eigenvalue is $I_{\boldsymbol{\chi}} = \mathrm{tr}\,\boldsymbol{\chi}$.

The conditions that the quadratic have a double zero root are $I_{\boldsymbol{\chi}} = 0$, $II_{\boldsymbol{\chi}} = 0$, or

$$\mathbf{C}^* \cdot \boldsymbol{\Phi}\mathbf{C}^* = 0, \qquad \mathbf{C}^* \cdot \mathbb{B}\mathbf{C}^* = 0, \tag{A.17}$$

where $\boldsymbol{\Phi}$ is the real positive definite tensor given by

$$\boldsymbol{\Phi} = (\mathrm{tr}\,\mathbb{B}^{-1})\mathbf{1} - \mathbb{B}^{-1}. \tag{A.18}$$

In general, any central plane will cut the ellipsoids associated with $\boldsymbol{\Phi}$ and with $\mathbb{B}$ in a pair of concentric ellipses. There are two 'exceptional' central planes for which each of these ellipses is similar and similarly situated to the other ellipse [11]. The two bivectors $\mathbf{C}^*$ which satisfy (A.17) may be obtained by choosing $\mathbf{C}^*$ to lie in an exceptional central plane such that its ellipse is similar and similarly situated to the elliptical section of the $\boldsymbol{\Phi}$ or $\mathbb{B}$ ellipsoid by the exceptional plane. In general, there are just two bivectors $\mathbf{C}^*$ which satisfy (A.17). Thus, in general, there are just two bivectors $\mathbf{C}^*$ for which $\boldsymbol{\chi}$ has a triple zero root.

A.3. Eigenbivectors of $\boldsymbol{\chi}$

How to proceed to determine the eigenbivectors of $\boldsymbol{\chi}$ will depend upon whether or not $\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+$ and $\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^-$ are zero. Accordingly, we consider the four cases:

Case (i): $\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+ \neq 0$, $\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^- \neq 0$.
Case (iia): $\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+ = 0$, $\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^- \neq 0$.
Case (iib): $\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+ \neq 0$, $\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^- = 0$.
Case (iii): $\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+ = 0$, $\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^- = 0$.

Case (i): $\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+ \neq 0$, $\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^- \neq 0$. The form of (A.9) immediately gives the eigenbivectors of $\boldsymbol{\chi}$. We have

$$\boldsymbol{\chi}\mathbf{A}^\pm = \delta^\pm \mathbf{A}^\pm, \tag{A.19}$$

where

$$\delta^\pm = \lambda_2^{-2} - \frac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})[(\mathbf{n}^+ \cdot \boldsymbol{\Pi}\mathbf{n}^+)(\mathbf{n}^- \cdot \boldsymbol{\Pi}\mathbf{n}^-)]^{1/2}(\mathbf{K}^+ \cdot \mathbf{K}^- \pm 1),$$
$$\mathbf{A}^\pm = \frac{\mathbf{K}^+ \pm \mathbf{K}^-}{[2(1 + \mathbf{K}^+ \cdot \mathbf{K}^-)]^{1/2}}. \tag{A.20}$$

We note that [13]

$$\boldsymbol{\Pi}\mathbf{A}^{\pm} = \mathbf{A}^{\pm}, \qquad \mathbf{A}^{\pm}\cdot\mathbf{A}^{\pm} = 1,$$
$$\mathbf{A}^{+}\cdot\mathbf{A}^{-} = 0, \qquad \mathbf{A}^{+}\cdot\mathbb{B}^{-1}\mathbf{A}^{-} = (\boldsymbol{\Pi}\mathbf{A}^{+})\cdot\mathbb{B}^{-1}(\boldsymbol{\Pi}\mathbf{A}^{-}) = \mathbf{A}^{+}\cdot\boldsymbol{\chi}\mathbf{A}^{-} = 0. \tag{A.21}$$

We also note, on using (A.20) and (A.21), that

$$\delta^{+} - \delta^{-} = -(\lambda_3^{-2} - \lambda_1^{-2})[(\mathbf{n}^{+}\cdot\boldsymbol{\Pi}\mathbf{n}^{+})(\mathbf{n}^{-}\cdot\boldsymbol{\Pi}\mathbf{n}^{-})]^{1/2},$$
$$\delta^{+} + \delta^{-} = \mathbf{A}^{+}\cdot\boldsymbol{\chi}\mathbf{A}^{+} + \mathbf{A}^{-}\cdot\boldsymbol{\chi}\mathbf{A}^{-} = \mathbf{A}^{+}\cdot\mathbb{B}^{-1}\mathbf{A}^{+} + \mathbf{A}^{-}\cdot\mathbb{B}^{-1}\mathbf{A}^{-} = \operatorname{tr}\mathbb{B}^{-1} - \mathbf{C}^{*}\cdot\mathbb{B}^{-1}\mathbf{C}^{*}$$
$$= \left(\frac{II}{III}\right) - \mathbf{C}^{*}\cdot\mathbb{B}^{-1}\mathbf{C}^{*}, \qquad \delta^{+}\delta^{-} = (\mathbf{A}^{+}\cdot\mathbb{B}^{-1}\mathbf{A}^{+})(\mathbf{A}^{-}\cdot\mathbb{B}^{-1}\mathbf{A}^{-}) = \frac{\mathbf{C}^{*}\cdot\mathbb{B}\mathbf{C}^{*}}{III}. \tag{A.22}$$

Thus, the eigenbivectors of $\boldsymbol{\chi}$ are $\mathbf{A}^{\pm}$, with corresponding distinct eigenvalues $\delta^{\pm}$, and $\mathbf{C}$ with corresponding eigenvalue zero.

Remark. Special case $\mathbf{C}\cdot\mathbf{j} = 0$.

We note that

$$\boldsymbol{\Pi}\mathbf{n}^{+}\wedge\boldsymbol{\Pi}\mathbf{n}^{-} = \frac{[\mathbf{C}\wedge(\mathbf{n}^{+}\wedge\mathbf{C})]\wedge[\mathbf{C}\wedge(\mathbf{n}^{-}\wedge\mathbf{C})]}{(\mathbf{C}\cdot\mathbf{C})^2} = [(\mathbf{n}^{+}\wedge\mathbf{n}^{-})\cdot\mathbf{C}^{*}]\mathbf{C}^{*} = 2\alpha\gamma(\mathbf{C}^{*}\cdot\mathbf{j})\mathbf{C}^{*}. \tag{A.23}$$

Assuming that neither $\boldsymbol{\Pi}\mathbf{n}^{+}$ nor $\boldsymbol{\Pi}\mathbf{n}^{-}$ is zero, it follows that it is only when $\mathbf{C}^{*}\cdot\mathbf{j} = 0$ that $\boldsymbol{\Pi}\mathbf{n}^{+}$ and $\boldsymbol{\Pi}\mathbf{n}^{-}$ are parallel. Assuming $\mathbf{C}^{*}\cdot\mathbf{j} = 0$, then

$$\boldsymbol{\Pi}\mathbf{n}^{+} = (\alpha C_3^{*} - \gamma C_1^{*})(C_3^{*}\mathbf{i} - C_1^{*}\mathbf{k}), \qquad \boldsymbol{\Pi}\mathbf{n}^{-} = (\alpha C_3^{*} + \gamma C_1^{*})(C_3^{*}\mathbf{i} - C_1^{*}\mathbf{k}). \tag{A.24}$$

We digress to consider the possibility that $\alpha C_3^{*} = \gamma C_1^{*}$, in which case $\boldsymbol{\Pi}\mathbf{n}^{+} = \mathbf{0}$. Then, using $C_2^{*} = 0$, it follows that $\mathbf{C}^{*} = C_1^{*}\mathbf{n}^{+}/\alpha$ so that $\mathbf{C}^{*}$ is a 'real' bivector, that is, a scalar multiple of a real vector. We exclude consideration of this possibility because it leads to homogeneous waves. Similarly, we exclude consideration of the possibility that $\alpha C_3^{*} = -\gamma C_1^{*}$ so that $\boldsymbol{\Pi}\mathbf{n}^{-} = \mathbf{0}$ and using $C_2^{*} = 0$, leads to $\mathbf{C}^{*} = C_1^{*}\mathbf{n}^{-}/\alpha$, again a real propagation bivector, leading to homogeneous waves.

Returning now to (A.24) with $\alpha^2(C_3^{*})^2 \neq \gamma^2(C_1^{*})^2$, it follows from (A.9) that

$$\boldsymbol{\chi} = [\lambda_2^{-2} - (\lambda_3^{-2} - \lambda_1^{-2})(\alpha^2(C_3^{*})^2 - \gamma^2(C_1^{*})^2)]\mathbf{K}^{+}\otimes\mathbf{K}^{+} + \lambda_2^{-2}\mathbf{j}\otimes\mathbf{j},$$
$$\mathbf{K}^{+} = \frac{\boldsymbol{\Pi}\mathbf{n}^{+}}{\alpha C_3^{*} - \gamma C_1^{*}} = C_3^{*}\mathbf{i} - C_1^{*}\mathbf{k}. \tag{A.25}$$

The eigenbivectors of $\boldsymbol{\chi}$ are now $\mathbf{C}^{*}$, $\mathbf{K}^{+}$, and $\mathbf{j}$, with corresponding eigenvalues 0, $\lambda_2^{-2} - (\lambda_3^{-2} - \lambda_1^{-2})(\alpha^2(C_3^{*})^2 - \gamma^2(C_1^{*})^2)$, and λ_2^{-2}. Because $C_2^{*} = 0$ and $(C_1^{*})^2 + (C_3^{*})^2 = 1$, it follows that equating any two of these eigenvalues will lead to a $\mathbf{C}^{*}$ which is a real bivector. So, isotropic eigenbivectors are only possible when the propagation bivector $\mathbf{C}$ is a real bivector. Accordingly, only homogeneous circularly polarized waves are possible when $C_2^{*} = 0$.

Thus, this special case $\mathbf{C}\cdot\mathbf{j} = 0$ plays no role in the study of circularly polarized inhomogeneous plane waves.

Case (iia): $\mathbf{n}^{+}\cdot\boldsymbol{\Pi}\mathbf{n}^{+} = 0$, $\mathbf{n}^{-}\cdot\boldsymbol{\Pi}\mathbf{n}^{-} \neq 0$. In this case $(\boldsymbol{\Pi}\mathbf{n}^{+})\cdot(\boldsymbol{\Pi}\mathbf{n}^{+}) = 0$, so that $\boldsymbol{\Pi}\mathbf{n}^{+}$ is an isotropic bivector. Indeed, the orthogonal projection of the ellipse of $\mathbf{C}^{*}$ onto the plane with normal $\mathbf{n}^{+}$ is a circle. So

$$\boldsymbol{\chi}\boldsymbol{\Pi}\mathbf{n}^{+} = \nu_{\perp}\boldsymbol{\Pi}\mathbf{n}^{+}, \quad \nu_{\perp} = \lambda_2^{-2} - \tfrac{1}{2}(\lambda_3^{-2} - \lambda_1^{-2})\mathbf{n}^{+}\cdot\boldsymbol{\Pi}\mathbf{n}^{-}. \tag{A.26}$$

We also have $\boldsymbol{\chi}\mathbf{C} = \mathbf{0}$, $\boldsymbol{\Pi}\mathbf{C} = \mathbf{0}$. It may be checked that there is no bivector $\boldsymbol{\Pi}\mathbf{n}^{+} + \epsilon\boldsymbol{\Pi}\mathbf{n}^{-}$ ($\epsilon \neq 0$) which is orthogonal to the eigenbivector $\mathbf{C}$ and forming with $\boldsymbol{\Pi}\mathbf{n}^{+}$ a third eigenbivector of $\boldsymbol{\chi}$. Thus, $\boldsymbol{\Pi}\mathbf{n}^{+}$ is an isotropic eigenbivector of $\boldsymbol{\chi}$ with eigenvalue $\nu_{\perp}$.

Case (iib): $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ \neq 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$. This is similar to Case (iia). In this case, $\mathbf{\Pi n}^-$ is an isotropic eigenbivector of $\boldsymbol{\chi}$ with eigenvalue $\nu_\perp$. Also, of course, $\mathbf{\Pi n}^-$ is orthogonal to $\mathbf{C}$, the eigenbivector with eigenvalue zero.

Case (iii): $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$. Here both $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are isotropic bivectors—the ellipse of $\mathbf{C}^*$ when projected upon a plane with normal $\mathbf{n}^+$ is a circle, and so is its projection onto a plane with normal $\mathbf{n}^-$. Also, $\mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = 0$, $\mathbf{n}^- \cdot \mathbf{\Pi n}^- = 0$, lead to

$$\mathbf{C}^* \cdot (\mathbf{n}^+ \pm \mathbf{n}^-) = 0, \qquad (\mathbf{C}^* \cdot \mathbf{n}^\pm)^2 = 1, \tag{A.27}$$

so that the ellipse of $\mathbf{C}^*$ lies *either* in a plane with normal along $\mathbf{n}^+ + \mathbf{n}^-$ ($= 2\alpha\mathbf{i}$, recall (2.14)), which is along the internal bisector of the angle between $\mathbf{n}^+$ and $\mathbf{n}^-$, *or* in a plane with normal along $\mathbf{n}^+ - \mathbf{n}^-$ ($= 2\gamma\mathbf{k}$, recall (2.14)), which is along the external bisector of the angle between $\mathbf{n}^+$ and $\mathbf{n}^-$. Thus, there are only two cases:

$$\text{(a)} \quad \mathbf{C}^* \cdot \mathbf{i} = 0, \qquad \text{(b)} \quad \mathbf{C}^* \cdot \mathbf{k} = 0. \tag{A.28}$$

In case (a), we find (recall (2.14)) from (A.27),

$$\mathbf{C}^* = \frac{\pm \mathrm{i}\alpha\mathbf{j} + \mathbf{k}}{\gamma}, \qquad \mathbf{n}^+ \cdot \mathbf{\Pi n}^- = 2\alpha^2, \qquad \boldsymbol{\chi}\mathbf{\Pi n}^\pm = \lambda_1^{-2}\mathbf{\Pi n}^\pm. \tag{A.29}$$

Also, these isotropic eigenbivectors $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are linearly independent. Indeed, if $\mathbf{\Pi n}^+ = \mu\mathbf{\Pi n}^-$ for some scalar μ, then $0 = \mathbf{n}^+ \cdot \mathbf{\Pi n}^+ = \mu\mathbf{n}^+ \cdot \mathbf{\Pi n}^- = 2\mu\alpha^2$, so that $\mu = 0$.

In case (b), we find

$$\mathbf{C}^* = \frac{\mathbf{i} \pm \mathrm{i}\gamma\mathbf{j}}{\alpha}, \qquad \mathbf{n}^+ \cdot \mathbf{\Pi n}^- = 2\gamma^2, \qquad \boldsymbol{\chi}\mathbf{\Pi n}^\pm = \lambda_3^{-2}\mathbf{\Pi n}^\pm. \tag{A.30}$$

As before, the isotropic eigenbivectors of $\boldsymbol{\chi}$, $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are linearly independent. Thus, $\boldsymbol{\chi}$ has eigenvalues zero and double eigenvalue λ_1^{-2} (in case (a)), or λ_3^{-2} (in case (b)), and corresponding eigenbivector $\mathbf{C}$ and isotropic eigenbivectors $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$. Of course, $\mathbf{C}$, $\mathbf{\Pi n}^+$, and $\mathbf{\Pi n}^-$ are linearly independent. Indeed, if for some scalars p, q,

$$\mathbf{C}^* = p\mathbf{\Pi n}^+ + q\mathbf{\Pi n}^-, \tag{A.31}$$

then

$$\mathbf{0} = \mathbf{\Pi C}^* = p\mathbf{\Pi n}^+ + q\mathbf{\Pi n}^-, \tag{A.32}$$

so that $\mathbf{\Pi n}^+$ is a scalar multiple of $\mathbf{\Pi n}^-$. But we have already seen that $\mathbf{\Pi n}^+$ and $\mathbf{\Pi n}^-$ are linearly independent. Thus $\mathbf{C}$, $\mathbf{\Pi n}^+$, and $\mathbf{\Pi n}^-$ are linearly independent.

References

[1] J. Hadamard, Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique, Hermann, Paris, 1903.

[2] F. John, Plane elastic waves of finite amplitude. Hadamard materials and harmonic materials, Commun. Pure Appl. Math. 19 (1966) 309–341.

[3] P.K. Currie, M. Hayes, Longitudinal and transverse waves in finite elastic strain. Hadamard and Green materials, J. Inst. Math. Appl. 5 (1969) 140–161.

[4] R.W. Ogden, Waves in isotropic elastic materials of Hadamard, Green, or harmonic type, J. Mech. Phys. Solids 18 (1970) 149–163.

[5] Ph. Boulanger, M. Hayes, C. Trimarco, Finite-amplitude waves in deformed Hadamard elastic materials, Geophys. J. Int. 118 (1994) 447–458.

[6] M. Hayes, Inhomogeneous plane waves, Arch. Rat. Mech. Anal. 85 (1984) 41–79.

[7] M. Hayes, A remark on Hadamard materials, Quart. J. Mech. Appl. Math. 21 (1968) 141–146.

[8] A.J. Willson, Plate waves in Hadamard materials, J. Elasticity 7 (1968) 103–111.

[9] P. Chadwick, D.A. Jarvis, Surface waves in a pre-stressed elastic body, Proc. R. Soc. London A 366 (1979) 517–536.

[10] M. Destrade, Finite-amplitude inhomogeneous plane waves in a deformed Blatz–Ko material, in: Proceedings of the First Canadian Conference on Nonlinear Solid Mechanics, University Press, Victoria, 1999, pp. 89–98.
[11] Ph. Boulanger, M. Hayes, Bivectors and Waves in Mechanics and Optics, Chapman & Hall, London, 1993.
[12] Ph. Boulanger, M. Hayes, Propagating and static exponential solutions in a deformed Mooney–Rivlin material, in: M.M. Carroll, M. Hayes (Eds.), Nonlinear Effects in Fluids and Solids, Plenum Press, New York, 1996, pp. 113–123.
[13] M. Destrade, Inhomogeneous plane waves in deformed elastic materials, Ph.D. Thesis, University College, Dublin, 1999.
[14] M. Destrade, Finite-amplitude inhomogeneous plane waves in a deformed Mooney–Rivlin material, Quart. J. Mech. Appl. Math. 53 (2000) 343–361.