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Mechanics of Materials 35 (2003) 931–939
MECHANICS OF MATERIALS
www.elsevier.com/locate/mechmat

Rayleigh waves in symmetry planes of crystals: explicit secular equations and some explicit wave speeds

M. Destrade *

Laboratoire de Modélisation en Mécanique, UMR 7607, CNRS, Université Pierre et Marie Curie, Boite 162, 4 Place Jussieu, 75252 Paris Cedex 05, France

Received 21 March 2002; received in revised form 29 July 2002

Abstract

Rayleigh waves are considered for crystals possessing at least one plane of symmetry. The secular equation is established explicitly for surface waves propagating in any direction of the plane of symmetry, using two different methods. This equation is a quartic for the squared wave speed in general, and a biquadratic for certain directions in certain crystals, where it may itself be solved explicitly. Examples of such materials and directions are found in the case of monoclinic crystals with the plane of symmetry at $x_3 = 0$. The cases of orthorhombic materials and of incompressible materials are also treated.

1. Introduction

The simplest physical setting involving a boundary value problem for an elastic material is that of a semi-infinite body with a plane boundary left free of tractions. However, the consideration of small amplitude deformations and motions of such a half-space leads usually to considerable mathematical difficulties, especially when the material is anisotropic. Indeed in the case of general (triclinic) anisotropy, the equations of motion (or of equilibrium) lead to the resolution of a sextic for the partial inhomogeneous plane waves (or deformations), whose roots cannot be obtained explicitly. Consequently, closed-form solutions have been sought for materials with at least orthorhombic symmetry, because then the equations of motion lead to a biquadratic, and because this case covers 16 different types of common symmetry classes such as tetragonal, hexagonal, or cubic (Royer and Dieulesaint, 1984).

In between the classes of triclinic crystals (no plane of symmetry) and of orthorhombic crystals (three orthogonal planes of symmetry), is the class of monoclinic crystals, with only one plane of symmetry. Among the three possibilities for the orientation of the symmetry plane, the configuration of a half-space $x_2 \geqslant 0$ made of monoclinic material with the plane of symmetry at $x_3 = 0$ is particularly important for two-dimensional deformations: (a) because in-plane stress and in-plane strain decouple from anti-plane stress and anti-plane strain, respectively, so that the equations of motion yield a quartic; and (b) because

* Tel.: +33-1-44277166/44275259.
E-mail address: destrade@lmm.jussieu.fr (M. Destrade).

0167-6636/03/$ - see front matter
doi:10.1016/S0167-6636(02)00294-6

these materials are structurally invariant (Ting, 2000) that is, the stress–strain relationships retain their form with respect to rotations in the (x_1, x_2)-plane about the x_3-direction, so that results obtained along the material axes x_1, x_2, x_3, are easily transposed along the rotated axes x_1^*, x_2^*, x_3, say. On the other hand, the problem of surface Rayleigh waves is also of prime importance because it is relevant to the study of many other problems for anisotropic elastic half-spaces, such as: near-the-surface stability analysis of a deformed half-space (Biot, 1965), normal forces applied to a half-space (Lamb, 1904), punch and indentation of half-space (Green and Zerna, 1968), steady state crack propagation (Broberg, 1999), and so on. An up-to-date account on the research and applications of surface acoustic waves in materials science can be found in (Hess, 2002). In the present paper, the *secular equation* is established for surface waves in monoclinic crystals with the plane of symmetry at $x_3 = 0$, where by 'secular equation' is meant the function of the squared wave speed c^2 which is zero when the tractions on the plane $x_2 = 0$ and at $x_2 \to \infty$ are zero. This equation is valid for the propagation of a Rayleigh wave in any direction of a symmetry plane for crystals possessing one plane of symmetry and of course for crystals of a higher order symmetry class, such as orthorhombic symmetry. It is also easy to take an eventual incompressibility of the elastic half-space into account (Destrade et al., 2002). Finally, after a rotation about the x_3-axis, the quartic secular equation may reduce to a biquadratic which can then be solved explicitly.

The secular equation is obtained in Section 3, after the equations of motion and boundary conditions for the problem have been recalled in Section 2. This equation is obtained in two different manners, first in "covariant" (Furs, 1997) form, then as a quartic in the squared wave speed (Currie, 1979; Destrade, 2001; Ting, 2002a). In the final section (Section 4), the results are applied to other situations. First, a rotation is made for the (x_1, x_2) plane about the x_3-axis and, at least for three monoclinic crystals (diallage, gypsum and tin fluoride), two directions are found for which the secular equation may be solved explicitly. Then the results are specialized from monoclinic to orthorhombic symmetry. Finally, the constraint of incompressibility is taken into account, and a numerical problem left open by Nair and Sotiropoulos (1999) is resolved.

Throughout the paper, the dynamical analysis is based on the use of the components of the tractions rather than the displacements, and expressions are found in terms of the stiffnesses as well as in terms of the reduced compliances.

2. Preliminaries

Here we recall the equations of motion for a linearly elastic semi-infinite body, made of a monoclinic material with the plane of symmetry at $x_3 = 0$, and seek a solution in the form of a surface wave solution, that is a solution which propagates in the x_1-direction, leaves the plane $x_2 = 0$ free of tractions, and vanishes as $x_2 \to \infty$. Because for such materials, in-plane motions are decoupled from anti-plane motions (Stroh, 1962), it is sufficient to seek a solution in the form of a two-component displacement vector $\mathbf{u}$, such as

$$\mathbf{u}(x_1, x_2, t) = [U_1(x_2), U_2(x_2), 0]^{\mathrm{T}} e^{ik(x_1 - ct)}, \tag{1}$$

where U_1, and U_2 are functions of x_2 satisfying $U_1(\infty) = U_2(\infty) = 0$, k is the wave number, and c is the wave speed.

With this convention, the equations of motion are written as (Mozhaev, 1995)

$$\boldsymbol{\alpha}\mathbf{U}'' + i\boldsymbol{\beta}\mathbf{U}' - \boldsymbol{\gamma}\mathbf{U} = \mathbf{0}, \tag{2}$$

where $\mathbf{U} = [U_1, U_2]^{\mathrm{T}}$ and the prime denotes differentiation with respect to kx_2. Here the symmetric 2×2 matrices α_{ij}, β_{ij}, and γ_{ij}, are given in terms of the elastic stiffnesses C's and of the mass density ρ by

$$\boldsymbol{\alpha} = \begin{bmatrix} C_{66} & C_{26} \\ C_{26} & C_{22} \end{bmatrix},$$
$$\boldsymbol{\beta} = \begin{bmatrix} 2C_{16} & C_{12} + C_{66} \\ C_{12} + C_{66} & 2C_{26} \end{bmatrix},$$
$$\boldsymbol{\gamma} = \begin{bmatrix} C_{11} - \rho c^2 & C_{16} \\ C_{16} & C_{66} - \rho c^2 \end{bmatrix}. \tag{3}$$

Finally for the problem at hand, the following boundary conditions must also be satisfied:

$$C_{66}U_1'(0) + C_{26}U_2'(0) + \mathrm{i}C_{16}U_1(0) + \mathrm{i}C_{66}U_2(0) = 0,$$
$$C_{26}U_1'(0) + C_{22}U_2'(0) + \mathrm{i}C_{12}U_1(0) + \mathrm{i}C_{26}U_2(0) = 0. \tag{4}$$

Dual to this approach is one involving the components of the tractions acting upon the planes parallel to the free surface, instead of the components of the mechanical displacement. Indeed, just as in-plane strain is decoupled from anti-plane strain, so is in-plane stress from anti-plane stress (Stroh, 1962; Ting, 1996; Destrade, 2001). Thus, introducing the scalars functions $t_1(x_2)$ and $t_2(x_2)$, defined by

$$\sigma_{21}(x_1, x_2, t) = t_1(x_2)\mathrm{e}^{\mathrm{i}k(x_1 - ct)},$$
$$\sigma_{22}(x_1, x_2, t) = t_2(x_2)\mathrm{e}^{\mathrm{i}k(x_1 - ct)}, \tag{5}$$

where σ_{21} and σ_{22} are the in-plane stress components, the equations of motions may be written as (Destrade, 2001)

$$\hat{\boldsymbol{\alpha}}\mathbf{t}'' - \mathrm{i}\hat{\boldsymbol{\beta}}\mathbf{t}' - \hat{\boldsymbol{\gamma}}\mathbf{t} = \mathbf{0}, \tag{6}$$

where $\mathbf{t} = [t_1, t_2]^{\mathrm{T}}$. Here the symmetric 2×2 matrices $\hat{\alpha}_{ij}$, $\hat{\beta}_{ij}$, and $\hat{\gamma}_{ij}$ are given in terms of the components of the stiffness matrix $\mathbf{C}$ (Destrade, 2001) or of the components of the reduced compliance matrix $\mathbf{s}'$ (Ting, 2002a) by

$$\hat{\boldsymbol{\alpha}} = \begin{bmatrix} \frac{1}{\eta - X} & 0 \\ 0 & -\frac{1}{X} \end{bmatrix}, \quad \hat{\boldsymbol{\beta}} = \begin{bmatrix} -2\frac{r_6}{\eta - X} & \frac{1}{X} - \frac{r_2}{\eta - X} \\ \frac{1}{X} - \frac{r_2}{\eta - X} & 0 \end{bmatrix},$$
$$\hat{\boldsymbol{\gamma}} = \begin{bmatrix} n_{66} + \frac{r_6^2}{\eta - X} - \frac{1}{X} & n_{26} + \frac{r_2 r_6}{\eta - X} \\ n_{26} + \frac{r_2 r_6}{\eta - X} & n_{22} + \frac{r_2^2}{\eta - X} \end{bmatrix}, \tag{7}$$

where $X = \rho c^2$ and

$$\varDelta = \begin{vmatrix} C_{22} & C_{26} \\ C_{26} & C_{66} \end{vmatrix} = C_{22}C_{66} - C_{26}^2,$$
$$\eta = \frac{1}{\varDelta}\begin{vmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{vmatrix} = \frac{1}{s'_{11}},$$
$$r_6 = -\frac{1}{\varDelta}\begin{vmatrix} C_{12} & C_{16} \\ C_{22} & C_{26} \end{vmatrix} = -\frac{s'_{16}}{s'_{11}},$$
$$r_2 = \frac{1}{\varDelta}\begin{vmatrix} C_{12} & C_{26} \\ C_{16} & C_{66} \end{vmatrix} = -\frac{s'_{12}}{s'_{11}},$$
$$n_{66} = \frac{C_{22}}{\varDelta} = \frac{1}{s'_{11}}\begin{vmatrix} s'_{11} & s'_{16} \\ s'_{16} & s'_{66} \end{vmatrix},$$
$$n_{22} = \frac{C_{66}}{\varDelta} = \frac{1}{s'_{11}}\begin{vmatrix} s'_{11} & s'_{12} \\ s'_{12} & s'_{22} \end{vmatrix},$$
$$n_{26} = -\frac{C_{26}}{\varDelta} = \frac{1}{s'_{11}}\begin{vmatrix} s'_{11} & s'_{16} \\ s'_{12} & s'_{26} \end{vmatrix}. \tag{8}$$

We recall that for two-dimensional deformations of a monoclinic material with the plane of symmetry at $x_3 = 0$ involving the coordinates x_1 and x_2 only, the relevant non-zero stiffnesses and reduced compliances are related through

$$\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}\begin{bmatrix} s'_{11} & s'_{12} & s'_{16} \\ s'_{12} & s'_{22} & s'_{26} \\ s'_{16} & s'_{26} & s'_{66} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

Finally the boundary conditions are written in a much simpler form than when displacement components are involved, as

$$t_1(0) = t_2(0) = 0, \quad \text{and} \quad t_1(\infty) = t_2(\infty) = 0. \tag{10}$$

3. The secular equation

3.1. The characteristic polynomial

Now we seek solutions to the equations of motion (6) in the form

$$\mathbf{t}(x_2) = \mathrm{e}^{\mathrm{i}kpx_2}\mathbf{T}, \tag{11}$$

where $\Im(p) > 0$, to ensure the decay of the wave amplitude away from the free surface, and $\mathbf{T}$ is a constant vector. So we have by (6),

$$\begin{bmatrix} -\hat{\alpha}_{11}p^2 + \hat{\beta}_{11}p - \hat{\gamma}_{11} & \hat{\beta}_{12}p - \hat{\gamma}_{12} \\ \hat{\beta}_{12}p - \hat{\gamma}_{12} & -\hat{\alpha}_{22}p^2 - \hat{\gamma}_{22} \end{bmatrix}\mathbf{T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12}$$

Hence, for nontrivial solutions to exist, p must be the root of a quartic, which corresponds to the determinant of the matrix above being equal to zero. This quartic, the *characteristic polynomial* of the equations of motion, may be written as

$$p^4 + 2\omega_3 p^3 + \omega_2 p^2 + 2\omega_1 p + \omega_0 = 0, \tag{13}$$

where the coefficients ω_3, ω_2, ω_1, and ω_0 are given by

$$\begin{aligned}
\omega_3 &= -\frac{s'_{16}}{s'_{11}},\\
\omega_2 &= \frac{1}{s'_{11}}[s'_{66} + 2s'_{12} - X(s'(1,2) + s'(1,6))],\\
\omega_1 &= -\frac{1}{s'_{11}}[s'_{26} + X(s'(1,2|2,6) - s'(1,6|1,2))],\\
\omega_0 &= \frac{1}{s'_{11}}[s'_{22} - X(s'(1,2) + s'(2,6)) + X^2 s'(1,2,6)].
\end{aligned} \tag{14}$$

Here, the expression $s'(n_1 \dots n_k|m_1 \dots m_k)$ represents the determinant of the $k \times k$ matrix which is a submatrix of the matrix s'_{ij} $(i,j = 1,\dots,6)$ and whose components correspond to the intersections of the rows $n_1,\dots,n_k$ and the columns $m_1,\dots,m_k$. Moreover, when $n_1 = m_1,\dots,n_k = m_k$, the shorter expression $s'(n_1 \dots n_k) \equiv s'(n_1 \dots n_k|n_1 \dots n_k)$ is used. The quartic (13) was obtained by Ting (2002a,b), and by Furs (1997) in terms of invariants of the stiffness matrix **C**. Note that when $X = \rho c^2 = 0$, the quartic of the elastostatic case is recovered as (Steeds, 1973, p. 72)

$$s'_{11}p^4 - 2s'_{16}p^3 + (2s'_{12} + s'_{66})p^2 - 2s'_{26}p + s'_{22} = 0. \tag{15}$$

Now we use the boundary conditions $(10)_{1,2}$ at the free surface to establish the secular equation. Let p_1 and p_2 be the roots of the characteristic polynomial (13) with positive imaginary part, and let $\mathbf{T}^{(r)}$ be a vector satisfying (12) when $p = p_r$ $(r = 1,2)$. These vectors are in the form, say,

$$\mathbf{T}^{(1)} = \begin{bmatrix} \hat\alpha_{22}p_1^2 + \hat\gamma_{22} \\ \hat\beta_{12}p_1 - \hat\gamma_{12} \end{bmatrix}, \quad \mathbf{T}^{(2)} = \begin{bmatrix} \hat\alpha_{22}p_2^2 + \hat\gamma_{22} \\ \hat\beta_{12}p_2 - \hat\gamma_{12} \end{bmatrix}. \tag{16}$$

Then, assuming $p_1 \neq p_2$, the tractions **t** defined in (5) are a combination of $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ for some constants q_1 and q_2,

$$\mathbf{t}(x_2) = q_1 e^{ikp_1x_2}\mathbf{T}^{(1)} + q_2 e^{ikp_2x_2}\mathbf{T}^{(2)}. \tag{17}$$

Using Ting's (2002a, 2002b) notation, **t** may be written as

$$\mathbf{t}(x_2) = \mathbf{B}\langle e^{ikx_*}\rangle\mathbf{q}, \tag{18}$$

where **q** is the vector $[q_1, q_2]^T$, and the matrices **B** and $\langle e^{ikx_*}\rangle$ are defined by

$$\mathbf{B} = [\mathbf{T}^{(1)}, \mathbf{T}^{(2)}], \quad \langle e^{ikx_*}\rangle = \mathrm{diag}(e^{ikp_1x_2}, e^{ikp_2x_2}). \tag{19}$$

The tractions satisfy the boundary conditions $(10)_1$, that is $\mathbf{t}(0) = \mathbf{0}$, when

$$\mathbf{Bq} = \mathbf{0}. \tag{20}$$

This system has non-trivial solutions when det $\mathbf{B} = 0$, that is when the following *secular equation* is satisfied

$$\hat\alpha_{22}\hat\beta_{12}p_1p_2 - \hat\alpha_{22}\hat\gamma_{12}(p_1 + p_2) - \hat\beta_{12}\hat\gamma_{22} = 0. \tag{21}$$

Now we try to obtain more satisfactory expressions for this equation.

3.2. The "covariant" secular equation

First, we decompose $p_1 + p_2$ and p_1p_2 into their real and imaginary parts as

$$p_1 + p_2 = u^+ + iu^-, \quad p_1p_2 = v^+ + iv^-. \tag{22}$$

It is known that when the roots of the quartic (13) are $p_1, p_2, \overline{p}_1$, and $\overline{p}_2$, then u^+, u^-, v^+, and v^- satisfy

$$\begin{aligned}
\omega_3 &= -u^+,\\
\omega_2 &= (u^+)^2 + (u^-)^2 + 2v^+,\\
\omega_1 &= -u^+v^+ - u^-v^-, \quad \omega_0 = (v^+)^2 + (v^-)^2,
\end{aligned} \tag{23}$$

which leads to the following cubic for v^+,

$$(v^+)^3 + b_2(v^+)^2 + b_1(v^+) + b_0 = 0, \tag{24}$$

where $b_2 = -\omega_2/2$, $b_1 = \omega_1\omega_3 - \omega_0$, and $b_0 = [\omega_0(\omega_2 - \omega_3^2) - \omega_1^2]/2$. On the other hand, the secular equation (21) may also be separated into its real and imaginary parts,

$$\begin{aligned}
&\hat\alpha_{22}\hat\beta_{12}(v^+) - \hat\alpha_{22}\hat\gamma_{12}(u^+) - \hat\beta_{12}\hat\gamma_{12} = 0,\\
&\hat\alpha_{22}\hat\beta_{12}(v^-) - \hat\alpha_{22}\hat\gamma_{12}(u^-) = 0.
\end{aligned} \tag{25}$$

At this point, it is important to emphasize that the system of six equations (23) and (25) for the five unknowns u^+, u^-, v^+, v^-, and X, is consistent. This is due a fundamental result of the modern theory of surface waves in anisotropic elasticity by Stroh (1962), which states that the complex secular equation (21) is actually equivalent to a single real equation (for alternative proofs see Currie, 1974; or Taylor, 1978). In fact, it can be proved that with the appropriate normalization of the Stroh eigenvectors, the real and imaginary parts of the secular equation are equal but opposite in sign (Ting, 1996, p. 469).

Now, v^+ must satisfy the cubic (24) and also the linear equation $(25)_1$, which using $(23)_1$, is

$$(v^+) + a_0 = 0,$$

where

$$a_0 = X\frac{s'_{16}[s'_{26} - Xs'(1,6|1,2)]}{s'_{11}[1 - X(s'_{11} - s'_{12})]} + X\frac{s'_{22} - s'(1,2)}{1 - Xs'_{11}}. \tag{26}$$

Working along similar lines but with the displacements components (1) rather than with the traction components (5), Furs (1997) showed that v^+ was simultaneously the root of a cubic (as in (24)) and of a quadratic (in contrast to (26)). By writing the resultant of those two polynomials, he obtained the secular equation in "covariant form" (his wording) as corresponding to the nullity of a 5×5 determinant. The same approach applied here to (24) and (26), would yield the "covariant" secular equation as corresponding to the nullity of a 4×4 determinant. However, v^+ is easily deduced from (26) as $v^+ = -a_0$ and substituted into (24) to yield the *"covariant" secular equation*,

$$a_0^3 - b_2 a_0^2 + b_1 a_0 - b_0 = 0. \tag{27}$$

Although not stated as such, Furs's "covariant" secular equation is a polynomial of degree 6 in $X = \rho c^2$. Eq. (27) is a polynomial of degree 9 in $X = \rho c^2$, but it may be factorized into the product of two polynomials, one of degree 5, which corresponds to spurious roots, and one of degree 4, which is the *quartic secular equation*. This very equation is obtained in a different and more direct manner in the next subsection as Eq. (31).

3.3. The quartic secular equation

In order to obtain the quartic secular equation directly, we recall that the vectors $\mathbf{T}^{(1)}$, $\mathbf{T}^{(2)}$, in (16) were computed using the second line of (12). When the first line is used, these vectors are

$$\mathbf{T}^{(1)} = \begin{bmatrix} \hat{\beta}_{12}p_1 - \hat{\gamma}_{12} \\ \hat{\alpha}_{11}p_1^2 - \hat{\beta}_{11}p_1 + \hat{\gamma}_{11} \end{bmatrix}, \quad \mathbf{T}^{(2)} = \begin{bmatrix} \hat{\beta}_{12}p_2 - \hat{\gamma}_{12} \\ \hat{\alpha}_{11}p_2^2 - \hat{\beta}_{11}p_2 + \hat{\gamma}_{11} \end{bmatrix}. \tag{28}$$

By the same steps that lead from (17) to (21), we obtain the following alternative form of the complex secular equation

$$\hat{\alpha}_{11}\hat{\beta}_{12}p_1p_2 - \hat{\alpha}_{11}\hat{\gamma}_{12}(p_1 + p_2) + \hat{\beta}_{11}\hat{\gamma}_{12} - \hat{\beta}_{12}\hat{\gamma}_{11} = 0. \tag{29}$$

By simple comparison of (21) and (29), the complex terms involving p_1p_2 and $p_1 + p_2$ may be eliminated, and the secular equation becomes real,

$$\hat{\alpha}_{11}\hat{\beta}_{12}\hat{\gamma}_{22} + \hat{\alpha}_{22}\hat{\beta}_{11}\hat{\gamma}_{12} - \hat{\alpha}_{22}\hat{\beta}_{12}\hat{\gamma}_{11} = 0. \tag{30}$$

This equation is a quartic in $X = \rho c^2$. It was established by this author (Destrade, 2001) in terms of the stiffnesses and by Ting (2002a) in terms of the compliances (see also Currie (1979) for a less explicit expression, obtained by writing the equations of motion for the displacements instead of the traction components). This polynomial of degree 4 in $X = \rho c^2$ is also the one obtained in the previous subsection by factorization of the "covariant" equation (27). Note that it may be obtained directly by using an adequate combination of the vectors (16) and (28) for the columns of the matrix $\mathbf{B}$. It is written explicitly with the coefficients in terms of the reduced compliances as

$$d_4X^4 + d_3X^3 + d_2X^2 + d_1X - 1 = 0, \tag{31}$$

where

$$\begin{aligned} d_4 &= s'_{11}(s'_{66}s'_{12}s'_{11} - s'_{66}s'^2_{11} + s'_{12}s'^2_{16} - 2s'_{16}s'_{11}s'_{26} \\ &\quad - s'_{12}s'_{11}s'_{22} + s'^2_{16}s'_{11} + s'^3_{12} - s'_{11}s'^2_{12} + s'^2_{11}s'_{22}), \\ d_3 &= -2s'^2_{11}s'_{22} + s'^3_{11} - s'^2_{11}s'_{12} + 3s'_{66}s'^2_{11} + s'^2_{12}s'_{11} \\ &\quad - 2s'^2_{16}s'_{11} + 4s'_{16}s'_{11}s'_{26} - 2s'_{66}s'_{11}s'_{12} + s'_{22}s'_{11}s'_{12} \\ &\quad - s'^2_{16}s'_{12}, \\ d_2 &= -3s'_{11}s'_{66} + s'^2_{16} - 2s'_{16}s'_{26} - 3s'^2_{11} + s'_{11}s'_{22} \\ &\quad + 2s'_{11}s'_{12} + s'_{66}s'_{12}, \\ d_1 &= 3s'_{11} - s'_{12} + s'_{66}. \end{aligned} \tag{32}$$

Values of the Rayleigh wave speed c_R are given for the 12 crystals of Table 1 in (Destrade, 2001). They correspond to the square root of the least positive root of (31).

In the isotropic case, the reduced compliances take the following values:

$$s'_{11} = s'_{22} = \frac{c_P^2}{4\rho c_S^2(c_P^2 - c_S^2)},$$
$$s'_{12} = \frac{2c_S^2 - c_P^2}{4\rho c_S^2(c_P^2 - c_S^2)}, \quad s'_{66} = \frac{1}{\rho c_S^2}, \tag{33}$$

and $s'_{16} = s'_{26} = 0$, where c_P and c_S are the speeds of the longitudinal and transverse bulk waves, respectively. Then the quartic (31) factorizes into the product of a polynomial of degree one and of the cubic found by Rayleigh (1885)

$$\left(\frac{c^2}{c_S^2}\right)^3 - 8\left(\frac{c^2}{c_S^2}\right)^2 + \left(24 - \frac{16c_S^2}{c_P^2}\right)\frac{c^2}{c_S^2} - 16\left(1 - \frac{c_S^2}{c_P^2}\right) = 0. \tag{34}$$

4. Applications

Now we apply the results of the previous section to other settings, namely, to the case of a surface wave propagating in any direction in the plane of symmetry $x_3 = 0$ for monoclinic crystals; then to the case of a surface wave propagating along a material axis for rhombic crystals; and finally to the case where the half-space is made of an incompressible monoclinic material. We also seek exact analytic solutions for the speed of Rayleigh waves.

4.1. Rotation in the plane of symmetry and explicit wave speeds

Monoclinic materials with the plane of symmetry at $x_3 = 0$ have a stiffness matrix $\mathbf{C}$ which is 'structurally invariant' that is, a matrix whose components which are zero remain zero after a rotation of the coordinate axes around the x_3-axis (Bond, 1943). Ting (2000) proved recently that the submatrix of the reduced compliance matrix appearing in (9) is also structurally invariant. Here, we exploit this property in order to derive the secular equation for surface waves propagating in any direction in the plane of symmetry. Previous efforts covering this topic include those of Chadwick and Wilson (1992) and of Furs (1997).

First, we consider a surface wave propagating on the plane $x_2^* = 0$ and polarized in the x_1^*-direction, where the coordinate system x_i^* is obtained from the material axes coordinate system x_i through a rotation about the x_3-axis by an arbitrary angle θ, say. Hence,

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$
$$m = \cos\theta, \quad n = \sin\theta. \tag{35}$$

Next, following Ting (2000), we infer that all the results from the previous section are directly applicable to this wave, as long as the reduced compliances s'_{ij} $(i, j = 1,2,6)$ are replaced by the following 'starred' quantities,

$$\begin{aligned}
(s'_{11})^* &= s'_{11}m^4 + (2s'_{12} + s'_{66})m^2n^2 + s'_{22}n^4 \\
&\quad + 2(s'_{16}m^2 + s'_{26}n^2)mn, \\
(s'_{22})^* &= s'_{22}m^4 + (2s'_{12} + s'_{66})m^2n^2 + s'_{11}n^4 \\
&\quad - 2(s'_{26}m^2 + s'_{16}n^2)mn, \\
(s'_{12})^* &= s'_{12} + (s'_{11} + s'_{22} - 2s'_{12} - s'_{66})m^2n^2 \\
&\quad - (s'_{16} - s'_{26})(m^2 - n^2)mn, \\
(s'_{16})^* &= s'_{16}m^4 - s'_{26}n^4 - 3(s'_{16} - s'_{26})m^2n^2 - [2s'_{11}m^2 \\
&\quad - 2s'_{22}n^2 - (2s'_{12} + s'_{66})(m^2 - n^2)]mn, \\
(s'_{26})^* &= s'_{26}m^4 - s'_{16}n^4 + 3(s'_{16} - s'_{26})m^2n^2 + [2s'_{22}m^2 \\
&\quad - 2s'_{11}n^2 - (2s'_{12} + s'_{66})(m^2 - n^2)]mn, \\
(s'_{66})^* &= s'_{66} + 4(s'_{11} + s'_{22} - 2s'_{12} - s'_{66})m^2n^2 \\
&\quad - 4(s'_{16} - s'_{26})(m^2 - n^2)mn.
\end{aligned} \tag{36}$$

In particular, the secular equation for the surface wave is the starred version of (31), that is $d_4^*X^4 + d_3^*X^3 + d_2^*X^2 + d_1^*X - 1 = 0$.

Because the coefficients d_4^*, d_3^*, d_2^*, and d_1^* in this quartic are functions of θ, it might be possible that for certain angles, the quartic turns into a biquadratic, for which the real root X may be found explicitly. Some work has been devoted to the search of explicit expressions for the speed of elastic surface waves. For instance, Lamb (1904) noted that Rayleigh's cubic equation (24) factorizes into the product of a polynomial of degree one and of a quadratic in c^2/c_S^2 when Poisson's ratio is 1/4, that is when $c_P^2 = 3c_S^2$ or equivalently, when the two Lamé constants are equal; in that case, $c_R^2 = 2c_S^2(1 - 1/\sqrt{3})$. In the general case, closed-form expressions are rather cumbersome for the relevant root of the cubic in isotropic (Nkemzi, 1997) or orthorhombic (Romeo, 2001) half-spaces and only approximate expressions are sought (Royer and Dieulesaint, 1984; Mozhaev, 1991). However, Mozhaev (1995) showed that for the special orthorhombic materials such that $c_{12} = c_{66}$ (or with an equivalent relationship for different choices of the material axes), the squared Rayleigh wave speed could be obtained as the root of a quadratic. Similarly, Ting (2002b) showed that the quartic (31) simplifies to the product of a squared polynomial of degree one and of quadratic in X for the special monoclinic materials with the plane of symmetry at $x_3 = 0$ such that $s'_{16} - 2s'_{26} = s'_{12} = 0$. Now the starred version of the quartic (31) may be rewritten in canonical form

$$Y^4 + aY^2 + bY + e = 0, \quad Y = X + \frac{d_3^*}{4d_4^*}, \tag{37}$$

where

$$\begin{aligned} a &= [8d_2^* d_4^* - 3(d_3^*)^2]/[8(d_4^*)^2], \\ b &= [(d_3^*)^3 - 4d_2^* d_3^* d_4^* + 8d_1^*(d_4^*)^2]/[8(d_4^*)^3], \\ e &= [16d_2^*(d_3^*)^2 d_4^* - 3(d_3^*)^4 - 256(d_4^*)^3 \\ &\quad - 64d_1^* d_3^*(d_4^*)^2]/[256(d_4^*)^4]. \end{aligned} \tag{38}$$

Clearly, it becomes a biquadratic if $b = 0$ for a certain angle $\theta = \alpha$ say. Then, solving the equation $b = 0$ at $\theta = \alpha$ for d_2^α, for instance, yields the following biquadratic,

$$Y^4 + 2\left[\frac{d_1^\alpha}{d_3^\alpha} - \left(\frac{d_3^\alpha}{4d_4^\alpha}\right)^2\right]Y^2 + \left(\frac{d_3^\alpha}{4d_4^\alpha}\right)^4 - \frac{d_1^\alpha d_3^\alpha}{8(d_4^\alpha)^2} - \frac{1}{d_4^\alpha} = 0, \tag{39}$$

whose explicit relevant root $X = \rho c_R^2 = Y - d_3^\alpha/(4d_4^\alpha)$ is

$$X = -\frac{d_3^\alpha}{4d_4^\alpha} + \sqrt{\left(\frac{d_3^\alpha}{4d_4^\alpha}\right)^2 - \frac{d_1^\alpha}{d_3^\alpha} - \sqrt{\left(\frac{d_1^\alpha}{d_3^\alpha}\right)^2 + \frac{1}{d_4^\alpha}}}. \tag{40}$$

Twice, the resolution allowed for a plus or a minus sign. One time, the plus sign was selected because $X = \rho c_R^2$ must be positive; the other time, the minus sign was selected by continuity with the known result (Lamb, 1904) in the special isotropic case where Poisson's ratio is 1/4.

Of course, the existence of an angle $\theta = \alpha$ such that $b = 0$ is not guaranteed. However, numerical simulations show that at least for diallage, gypsum, and tin fluoride, two angles α may indeed be found such that a surface wave propagating in the x_1^α-direction with attenuation in the x_2^α-direction, where (x_1^α, x_2^α) are obtained from the material axes (x_1, x_2) by a rotation about the x_3-axis of the angle α, has a velocity $c_R = \sqrt{X/\rho}$ which is given explicitly by (40). For diallage, the angles are $\alpha = 80.24°$ and $87.64°$ with corresponding Rayleigh speeds $c_R = 3960$ and 3952 m/s, respectively; for gypsum, the angles are $\alpha = 18.82°$ and $65.00°$ with corresponding Rayleigh speeds $c_R = 2895$ and 2946 m/s, respectively; and for tin fluoride, the angles are $\alpha = 4.01°$ and $31.21°$ with corresponding Rayleigh speeds $c_R = 1324$ and 1351 m/s, respectively.

4.2. Orthorhombic crystals

For orthorhombic crystals, $s'_{16} = s'_{26} = 0$, so that $\hat{\beta}_{11} = \hat{\gamma}_{12} = 0$ and the results obtained for monoclinic materials are greatly simplified. In particular, the characteristic polynomial (13) is now a biquadratic in p

$$p^4 - Sp^2 + P = 0, \tag{41}$$

where the real scalars S and P are given by

$$\begin{aligned} S &= \frac{1}{s'_{11}}[s'_{66} + 2s'_{12} - X(s'_{11}s'_{22} - s'^2_{12} + s'_{11}s'_{66})], \\ P &= \frac{1}{s'_{11}}(1 - Xs'_{66})[s'_{22} - X(s'_{11}s'_{22} - s'^2_{12})]. \end{aligned} \tag{42}$$

Note that, depending upon the sign of $S^2 - 4P$ and of S, the roots p_1 and p_2 of the biquadratic with positive imaginary parts are either purely imaginary:

$$p_1 = \mathrm{i}\sqrt{(-S + \sqrt{S^2 - 4P})/2},$$

$$p_2 = \mathrm{i}\sqrt{(-S - \sqrt{S^2 - 4P})/2}$$

(when $S^2 - 4P > 0,\ S < 0$), or of the form: $p_1 = a + \mathrm{i}b$, $p_2 = -a + \mathrm{i}b$, where

$$a = \sqrt{(S + 2\sqrt{P})/4}$$

and

$$b = \sqrt{(-S + 2\sqrt{P})/4}$$

(when $S^2 - 4P < 0$). In any case, we have

$$p_1 p_2 = -\sqrt{P}. \tag{43}$$

It is now easy to see that the *secular equation* (21) is simplified to (using (43))

$$\hat{\alpha}_{22}\sqrt{P} + \hat{\gamma}_{22} = 0. \tag{44}$$

Explicitly, the secular equation (44) is written as

$$\begin{aligned}(1 - Xs'_{11})&\sqrt{1 - Xs'_{66}} \\ &- X\sqrt{s'_{11}[s'_{22} - X(s'_{11}s'_{22} - s'^2_{12})]} \\ &= 0.\end{aligned} \tag{45}$$

The secular equation for surface waves in orthorhombic crystals was first established by Sveklo (1948) in terms of the elastic stiffnesses. Ting (2002b) found the cubic secular equation in terms of the elastic reduced compliances, an equation which may be deduced from (45) by rationalization; however, the squaring process introduces spurious roots, while the exact secular equation (45) has a unique root (Romeo, 2001).

Note that after rotation about the x_3-axis, the secular equation is again $d^*_4X^4 + d^*_3X^3 + d^*_2X^2 + d^*_1X - 1 = 0$, where the $d^{*\prime}s$ are given by (32) and (36) with $s'_{16} = s'_{26} = 0$. Also, special directions in which the secular equation is a biquadratic might also be found for orthorhombic crystals, following a procedure similar to the one exposed in the previous subsection.

4.3. Incompressible monoclinic materials

According to Klintworth and Stronge (1990), "many anisotropic composite materials appear relatively incompressible because their bulk modulus is large compared with their shear moduli. In particular, low-density cellular materials are highly compliant in shear because the flexural rigidity of the cell walls is small." Nair and Sotiropoulos (1997, 1999) also studied incompressible anisotropic materials; in particular they considered surface waves in monoclinic materials with the plane of symmetry at $x_3 = 0$, but did not obtain the secular equation explicitly.

Recently, Destrade et al. (2002) proved that in linear anisotropic elasticity, the constraint of incompressibility implied that certain relationships must be satisfied for some compliances. In the present context, the following relationships must hold,

$$s'_{11} + s'_{12} = s'_{12} + s'_{22} = s'_{16} + s'_{26} = 0, \tag{46}$$

and they greatly simplify the *quartic secular equation* (31) to

$$\begin{aligned}2s'^2_{11}(s'^2_{16} &- s'_{11}s'_{66})X^4 - 5s'_{11}(s'^2_{16} - s'_{11}s'_{66})X^3 \\ &+ [3s'^2_{16} - 4s'_{11}(s'_{11} + s'_{66})]X^2 \\ &+ (4s'_{11} + s'_{66})X - 1 = 0.\end{aligned} \tag{47}$$

This equation was obtained by Destrade et al. (2002) in a less explicit manner.

Nair and Sotiropoulos (1999) introduced the constants α, β, and γ defined by

$$\alpha = \frac{s'_{11}}{s'_{11}s'_{66} - s'^2_{16}}, \quad \beta = \frac{s'_{66}}{4s'_{11}} - 1, \quad \gamma = -\frac{s'_{16}}{s'_{11}}, \tag{48}$$

when these equations are solved for s'_{11}, s'_{16}, and s'_{66}, the secular equation (47) may be written as a quartic in $x \equiv X/\alpha = \rho c^2/\alpha$,

$$\begin{aligned}2x^4 &- 5(4\beta + 4 - \gamma^2)x^3 \\ &+ (16\beta + 20 - 3\gamma^2)(4\beta + 4 - \gamma^2)x^2 \\ &- 4(\beta + 2)(4\beta + 4 - \gamma^2)^2x + (4\beta + 4 - \gamma^2)^3 \\ &= 0.\end{aligned} \tag{49}$$

Now a numerical example is given, as the surface wave speed is computed in the case (Nair and

Sotiropoulos, 1999) where $\beta = 0.3$ and $\delta = 0.1$. Then, the secular equation (49) is the quartic

$$2.0000x^4 - 25.950x^3 + 128.56x^2 - 247.81x + 139.80 = 0 \tag{50}$$

for which the least real root is $x = 0.94671$, in agreement with the likely limit of the 'iterative solution' given by Nair and Sotiropoulos (1999).

Acknowledgements

I thank T.C.T. Ting for his insightful explanations about the Stroh formalism, and V.G. Mozhaev for sharing his extensive knowledge of the literature on surface acoustic waves, in particular of highly relevant articles in Russian.

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