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Cite this article: Destrade M, Fu YB, Nobili A. 2016 Edge wrinkling in elastically supported pre-stressed incompressible isotropic plates. *Proc. R. Soc. A* **472**: 20160410. http://dx.doi.org/10.1098/rspa.2016.0410

Received: 27 May 2016
Accepted: 4 August 2016

Subject Areas:
mechanics

Keywords:
edge wrinkles, buckling, pre-stressed plate, foundation, incremental deformation

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Edge wrinkling in elastically supported pre-stressed incompressible isotropic plates

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The equations governing the appearance of flexural static perturbations at the edge of a semi-infinite thin elastic isotropic plate, subjected to a state of homogeneous bi-axial pre-stress, are derived and solved. The plate is incompressible and supported by a Winkler elastic foundation with, possibly, wavenumber dependence. Small perturbations superposed onto the homogeneous state of pre-stress, within the three-dimensional elasticity theory, are considered. A series expansion of the plate kinematics in the plate thickness provides a consistent expression for the second variation of the potential energy, whose minimization gives the plate governing equations. Consistency considerations supplement a constraint on the scaling of the pre-stress so that the classical Kirchhoff–Love linear theory of pre-stretched elastic plates is retrieved. Moreover, a scaling constraint for the foundation stiffness is also introduced. Edge wrinkling is investigated and compared with body wrinkling. We find that the former always precedes the latter in a state of uni-axial pre-stretch, regardless of the foundation stiffness. By contrast, a general bi-axial pre-stretch state may favour body wrinkling for moderate foundation stiffness. Wavenumber dependence significantly alters the predicted behaviour. The results may be especially relevant to modelling soft biological materials, such as skin or tissues, or stretchable organic thin-films, embedded in a compliant elastic matrix.

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1. Introduction

The edge buckling phenomenon is ubiquitous in nature and it can be observed at the boundary of almost all biological thin structures. Examples include, among many, lettuce leaves, flower petals or the gut tube in animals, see the review by Li *et al.* [1]. There, the driving force behind edge wrinkling is undoubtedly growth. In Kabir *et al.* [2], growth-induced buckling of cell microtubules embedded in a compressed kinesin substrate is shown experimentally and it is modelled as instability of a beam-plate supported by a Winkler elastic foundation. Alongside biological systems, thin-film flexible materials, with special regard to organic films, easily develop stress-induced instability and, with it, a great potential for integrated applications in moving parts and complex geometries. In particular, stretchability (i.e. elasticity under tensile strain) and flexibility have been identified as key material properties required to develop collapsible and portable devices [3], bio-sensors and textile integration [4], energy scavengers [5] and embedded capacitors and batteries. Lipomi *et al.* [6] experimentally investigated pre-stressed stretchable organic photovoltaic cells laid on an elastic substrate as an application of body buckling to increase integration compliance in portable devices.

From a mechanical standpoint, modelling of edge buckling is related to edge wave propagation, whose consideration dates back to 1960 and is now credited to Konenkov [7], despite a long history of discovery and rediscovery, see the overview by Norris *et al.* [8] and the more recent contributions [9–13]. However, to move from edge waves to edge *wrinkles*, we must add the effects of a large enough pre-deformation such that the edge wave speed drops to zero. In this way, a localized static solution might exist in the neighbourhood of this pre-deformation.

Alongside some established mathematical tools, such as Gamma convergence and the asymptotic method [14], results which consistently separate flexural and extensional effects can be obtained through a Taylor expansion of the potential energy in powers of the plate thickness h, as in [15–17]. Kaplunov *et al.* [18] used an asymptotic technique to analyse vibrations of thin elastic pre-stressed incompressible plates in the low-frequency limit $\eta = kh \ll 1$, where k is the wavenumber, for the special case of plane deformation. Pichugin & Rogerson [19] provided an extension to the three-dimensional case. Tovstik [20] considered the vibrations of a pre-stressed transversely isotropic infinite thin plate that is supported by an elastic foundation with inertial contribution. Remarkably, no consistent attempt at considering edge wrinkles in pre-stressed elastic plates can be traced in the literature, to the best of the authors' knowledge.

In this paper, we derive the equations governing edge wrinkling of a homogeneously pre-stressed plate made of incompressible isotropic hyperelastic material, together with the corresponding boundary conditions, when the plate is bilaterally supported by a Winkler elastic foundation [21,22]. As in [17], we adopt a through-the-thickness expansion for the second variation of the plate energy, with the differences that our material is incompressible and the plate is elastically supported. These assumptions are introduced to better model soft solids and thin-films embedded in an elastic matrix. It is worth emphasizing that consideration of different scalings for the pre-stress $\boldsymbol{\sigma}$ leads to a diverse mechanical response. In this paper, we assume that the pre-stress is small as it scales as h^2 and the plate is thin, i.e. $h \ll 1$. Besides, the Winkler foundation is soft and its stiffness κ scales as h^3. As a result, a flexural behaviour for the supported plate is considered. By contrast, in [18,19] attention is set on a large pre-stress, for $\boldsymbol{\sigma} = \mathcal{O}(1)$ independently of h. Consequently, in-plane deformation (membrane regime) takes over. Indeed, we show that the scaling assumed for the pre-stress determines the leading term in the energy expansion, while consistency considerations suggest the proper scaling for the foundation stiffness.

The paper is structured as follows. Section 2 introduces the problem and presents the variational framework. We carry out the through-the-thickness energy expansion in §3 and minimize the second variation of the potential energy in §4. The plate governing equation as well as the boundary conditions are given in §5. We draw a comparison with the classical Kirchhoff–Love theory of pre-stressed plates in §6. In §7, we seek solutions in the form of edge wrinkles and derive the corresponding bifurcation curve. Body wrinkling is considered in §8 and its occurrence is compared with that of edge wrinkling as a function of the foundation stiffness and wavenumber dependence. Finally, conclusions are drawn in §9.

2. Formulation of the problem

Consider a hyperelastic plate B occupying the region B, named *equilibrium configuration*, of the three-dimensional Euclidean space $\mathcal{E}$ and let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ denote a fixed orthonormal basis set for $\mathcal{E}$ along the axes $\{x_1, x_2, x_3\}$. The plate is incompressible and it has been homogeneously pre-deformed. The equilibrium configuration takes the form

$$\mathrm{B} = \omega \times [-\tfrac{1}{2}h, \tfrac{1}{2}h], \tag{2.1}$$

where $h > 0$ denotes the plate thickness and the region ω in the plane $x_3 = 0$ is named the *plate mid-plane*. Here, we assume that $\boldsymbol{e}_3$ is a principal axis for the homogeneous pre-deformation, while no such provision is taken for $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Hence, the plate is pre-deformed by the application of a constant Cauchy stress $\boldsymbol{\sigma}$ such that $\sigma_{13} = \sigma_{23} = 0$, whereas the other shear stress components are generally non-zero (see figure 1 for the case of a uni-axial stress). For simplicity, we further assume $\sigma_{33} = 0$.

Having been homogeneously pre-stretched, the plate undergoes a small incremental motion. Thus, the deformation reads

$$\boldsymbol{\chi} = \boldsymbol{\chi}^{(0)} + \epsilon \boldsymbol{\chi}^{(1)}, \tag{2.2}$$

where $\boldsymbol{\chi}^{(0)}$ is the homogeneous pre-stretch and $\epsilon \boldsymbol{\chi}^{(1)}$ the small incremental deformation, where $|\epsilon| \ll 1$. Let $\boldsymbol{F} = \operatorname{grad} \boldsymbol{\chi}^{(0)}$ be the homogeneous gradient of the pre-deformation.

In this paper, we focus on *flexural edge wrinkles* arising in a thin plate, where *thin* is to be understood in the sense that the plate thickness h is small compared with the wrinkle wavelength $\ell = 2\pi k^{-1}$, i.e. $kh \ll 1$. The plate is uniformly and bilaterally supported along $\boldsymbol{e}_3$ by an elastic Winkler foundation with stiffness $\kappa > 0$. Besides, to fix ideas, we assume that the foundation reaction is directly applied to the plate mid-plane, although this restriction will prove unnecessary. Consequently, the plate top and bottom faces are stress-free, i.e.

$$\boldsymbol{\sigma}\boldsymbol{e}_3 = \boldsymbol{0} \quad \text{at } x_3 = \pm\frac{h}{2}. \tag{2.3}$$

The gradient of the incremental displacement (deprived of the small parameter ϵ) is

$$\boldsymbol{\Gamma} = \operatorname{grad} \boldsymbol{\chi}^{(1)} = \chi^{(1)}_{i,j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{2.4}$$

Likewise, let the incremental nominal stress [23]

$$\boldsymbol{\Sigma} = \boldsymbol{\mathcal{A}}\boldsymbol{\Gamma} + p\boldsymbol{\Gamma} - \dot{p}\boldsymbol{I}, \tag{2.5}$$

with components

$$\Sigma_{ji} = \mathcal{A}_{jilk}\Gamma_{kl} + p\Gamma_{ji} - \dot{p}\delta_{ij}. \tag{2.6}$$

Here, $\boldsymbol{\mathcal{A}}$ is the fourth-order tensor of the *instantaneous* elastic moduli, which is endowed with the major symmetry property [24, eqn (2.10)]

$$\mathcal{A}_{ijkl} = \mathcal{A}_{klij}, \tag{2.7}$$

while the scalar p is a Lagrange multiplier due to the internal constraint of incompressibility and $\dot{p}$ is its increment. We recall the connections among the instantaneous elastic moduli [24, eqn (3.6)],

$$\mathcal{A}_{jikl} - \mathcal{A}_{ijkl} = (\sigma_{jk} + p\delta_{jk})\delta_{il} - (\sigma_{ik} + p\delta_{ik})\delta_{jl}, \tag{2.8}$$

whence, multiplying through by Γ_{lk} and summing over repeated indexes, we get

$$\Sigma_{ji} - \Sigma_{ij} = \sigma_{jk}\Gamma_{ik} - \sigma_{ik}\Gamma_{jk}. \tag{2.9}$$

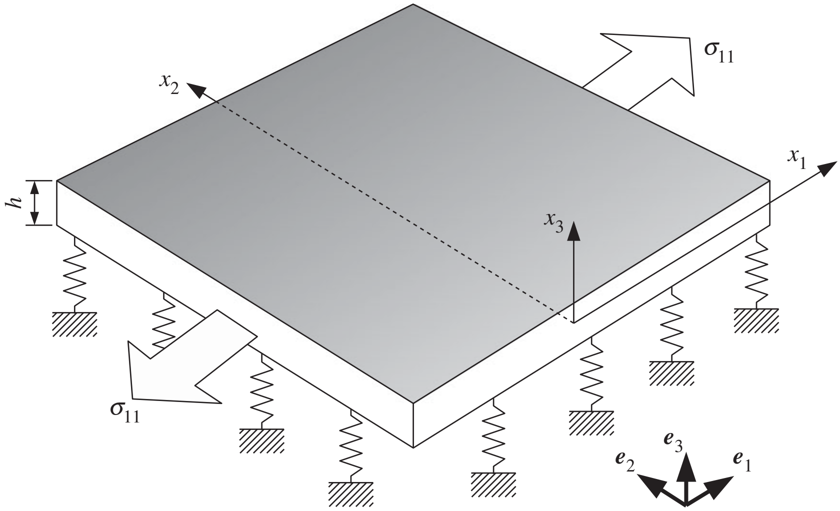

Figure 1. Hyperelastic semi-infinite plate resting on a Winkler foundation. Here the plate has been homogeneously pre-stretched along x_1 (equilibrium configuration) by the application of a uni-axial stress σ_{11}.

As the deformation $\boldsymbol{\chi}$ is a one-parameter family in ϵ, we can write the *potential energy* E of the system as a function of ϵ,

$$E(\epsilon) = \int_{\mathrm{B}} W \, \mathrm{d}V - \int_{\partial \mathrm{B}} \boldsymbol{t} \cdot \boldsymbol{\chi} \, \mathrm{d}S + E_{\mathrm{W}}(\epsilon), \tag{2.10}$$

where $W = W(\boldsymbol{F} + \epsilon \boldsymbol{\Gamma})$ is the elastic energy stored in B, $\boldsymbol{t}$ denotes the traction applied on ∂B, the boundary of B, and the last integral accounts for the contribution of the foundation. Following [15,17], when the potential energy $E(\epsilon)$ is expanded as a Taylor series about $\epsilon = 0$, the first variation vanishes because the current configuration is an equilibrium one. The second variation of the potential energy is

$$E''(0) = E''_{\mathrm{B}}(0) + E''_{\mathrm{W}}(0), \tag{2.11}$$

where the body contribution, $E''_{\mathrm{B}}(0)$, is developed in §4 while the Winkler foundation contribution is simply

$$E''_{\mathrm{W}}(0) = \int_{\omega} \kappa w^2 \, \mathrm{d}S, \tag{2.12}$$

where $\kappa > 0$ is named the *Winkler modulus* or the foundation stiffness.

3. Through-the-thickness expansions

Unless otherwise stated, the summation convention over twice repeated indexes is adopted, with the understanding that all Greek subscripts take on values in the set $\{1, 2\}$, while Roman subscripts range in the set $\{1, 2, 3\}$. A comma is used to denote partial differentiation with respect to the relevant coordinate, i.e. $w_{,1} = \partial w / \partial x_1$. We assume that the incremental fields admit the following through-the-thickness expansions in $x_3 \in [-h/2, h/2]$

$$\boldsymbol{\chi}^{(1)} = \boldsymbol{v} + w\boldsymbol{e}_3 + x_3 \boldsymbol{a} + \tfrac{1}{2} x_3^2 \boldsymbol{b} + \tfrac{1}{6} x_3^3 \boldsymbol{c} + \cdots \tag{3.1a}$$

and

$$\dot{p} = \dot{p}^{(0)} + x_3 \dot{p}^{(1)} + \tfrac{1}{2} x_3^2 \dot{p}^{(2)} + \cdots, \tag{3.1b}$$

where $\boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $w, \dot{p}^{(0)}, \dot{p}^{(1)}, \dot{p}^{(2)}$ are functions of x_1 and x_2 (see also [25,26]). Note that, at leading order, the displacement of the mid-plane has been decomposed as the in-plane displacement $v_\alpha \boldsymbol{e}_\alpha$, plus the transverse displacement $w\boldsymbol{e}_3$. Consequently, it may be assumed that $v_3 = 0$ without loss of generality.

For the gradient operator, we use the decomposition

$$\operatorname{grad} \boldsymbol{f} = \boldsymbol{f}_{,i} \otimes \boldsymbol{e}_i = \boldsymbol{f}_{,\alpha} \otimes \boldsymbol{e}_\alpha + \boldsymbol{f}_{,3} \otimes \boldsymbol{e}_3 = \nabla \boldsymbol{f} + \boldsymbol{f}_{,3} \otimes \boldsymbol{e}_3, \tag{3.2}$$

where ∇ denotes the two-dimensional nabla operator through which the two-dimensional divergence of a vector, $\nabla \cdot \boldsymbol{f} = f_{\alpha,\alpha}$, of a tensor, $(\nabla \cdot \boldsymbol{\Sigma})_j = \Sigma_{\alpha j,\alpha}$, and the two-dimensional gradient of a vector, $\nabla \boldsymbol{f} = f_{i,\beta} \boldsymbol{e}_i \otimes \boldsymbol{e}_\beta$, may be defined. Then, from (2.4) we obtain

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(0)} + x_3 \boldsymbol{\Gamma}^{(1)} + \tfrac{1}{2} x_3^2 \boldsymbol{\Gamma}^{(2)} + \cdots, \tag{3.3}$$

where

$$\boldsymbol{\Gamma}^{(0)} = \nabla \boldsymbol{v} + w_{,\alpha} \boldsymbol{e}_3 \otimes \boldsymbol{e}_\alpha + \boldsymbol{a} \otimes \boldsymbol{e}_3, \tag{3.4a}$$

$$\boldsymbol{\Gamma}^{(1)} = \nabla \boldsymbol{a} + \boldsymbol{b} \otimes \boldsymbol{e}_3 \tag{3.4b}$$

and

$$\boldsymbol{\Gamma}^{(2)} = \nabla \boldsymbol{b} + \boldsymbol{c} \otimes \boldsymbol{e}_3, \tag{3.4c}$$

or, in terms of components,

$$\Gamma^{(0)}_{\alpha\beta} = v_{\alpha,\beta}, \quad \Gamma^{(0)}_{\alpha 3} = a_\alpha, \quad \Gamma^{(0)}_{3\beta} = w_{,\beta}, \quad \Gamma^{(0)}_{33} = a_3, \tag{3.5a}$$

$$\Gamma^{(1)}_{\alpha\beta} = a_{\alpha,\beta}, \quad \Gamma^{(1)}_{\alpha 3} = b_\alpha, \quad \Gamma^{(1)}_{3\beta} = a_{3,\beta}, \quad \Gamma^{(1)}_{33} = b_3 \tag{3.5b}$$

and

$$\Gamma^{(2)}_{\alpha\beta} = b_{\alpha,\beta}, \quad \Gamma^{(2)}_{\alpha 3} = c_\alpha, \quad \Gamma^{(2)}_{3\beta} = b_{3,\beta}, \quad \Gamma^{(2)}_{33} = c_3. \tag{3.5c}$$

Upon substituting (3.3) into (2.5), we obtain

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{(0)} + x_3 \boldsymbol{\Sigma}^{(1)} + \tfrac{1}{2} x_3^2 \boldsymbol{\Sigma}^{(2)} + \ldots, \tag{3.6}$$

where, clearly,

$$\Sigma^{(0)}_{ji} = \mathcal{A}_{jilk} \Gamma^{(0)}_{kl} + p \Gamma^{(0)}_{ji} - \dot{p}^{(0)} \delta_{ij} \tag{3.7a}$$

and

$$\Sigma^{(1)}_{ji} = \mathcal{A}_{jilk} \Gamma^{(1)}_{kl} + p \Gamma^{(1)}_{ji} - \dot{p}^{(1)} \delta_{ij}, \tag{3.7b}$$

and so forth.

The stress-free boundary conditions at the top and bottom surfaces of the plate, equations (2.3), extend to the incremental stress and give

$$\Sigma^{(0)}_{3i} \pm \Sigma^{(1)}_{3i} \tfrac{1}{2} h + \Sigma^{(2)}_{3i} \tfrac{1}{8} h^2 \pm \mathcal{O}(h^3) = 0, \tag{3.8}$$

at $x_3 = \pm h/2$. Then, adding and subtracting together the two conditions, we get, to leading order [17, eqn (51)],

$$\Sigma^{(0)}_{3i} = -\Sigma^{(2)}_{3i} \tfrac{1}{8} h^2 \tag{3.9a}$$

and

$$\Sigma^{(1)}_{3i} = -\Sigma^{(3)}_{3i} \tfrac{1}{48} h^2. \tag{3.9b}$$

We observe that equation (3.8) shows that the assumption that the foundation reaction acts directly at the plate mid-plane may be abandoned with no harm provided that, as it will appear later, the foundation is soft and its reaction is $\mathcal{O}(h^3)$.

The incremental incompressibility condition, div $\boldsymbol{\chi}^{(1)}=0$, applied to the expansion (3.1*a*), gives

$$\nabla\cdot\boldsymbol{v}+x_3\nabla\cdot\boldsymbol{a}+\tfrac{1}{2}x_3^2\nabla\cdot\boldsymbol{b}+a_3+x_3b_3+\tfrac{1}{2}x_3^2c_3+\cdots=0 \tag{3.10}$$

and because the above expression needs to vanish for any value of x_3, the coefficients of this polynomial in x_3 vanish independently, i.e.

$$\nabla\cdot\boldsymbol{v}+a_3=0,\quad \nabla\cdot\boldsymbol{a}+b_3=0\quad\text{and}\quad \nabla\cdot\boldsymbol{b}+c_3=0\ldots. \tag{3.11}$$

We only consider flexural deformations, so we may set $v_\alpha=0$. It then follows, from the first of equations (3.11) that

$$a_3=0. \tag{3.12}$$

Now, taking $i=\alpha$ in (3.9*a*) and making use of equations (3.5*a*) and (3.7*a*), we obtain

$$\Sigma_{3\alpha}^{(0)}=\mathcal{A}_{3\alpha3\beta}a_\beta+(\mathcal{A}_{3\alpha\beta3}+p\delta_{\alpha\beta})w_{,\beta}=\mathcal{O}(h^2), \tag{3.13}$$

from which we deduce, with the help of equations (2.7), (2.8) and $\sigma_{33}=0$, that

$$a_\alpha=-w_{,\alpha}+\mathcal{O}(h^2). \tag{3.14}$$

To leading order, this result amounts to the well-known assumption in the Kirchhoff–Love plate theory of zero shear deformation along the cross-section, the latter remaining orthogonal to the mid-plane. Taking $i=3$ in (3.9*a*) yields $\dot{p}^{(0)}=\mathcal{O}(h^2)$ and it follows from equations (3.5*a*), (3.7*a*) and (3.14) that

$$\Sigma_{\alpha\beta}^{(0)}=\mathcal{O}(h^2). \tag{3.15}$$

Together, the second equation of (3.11) and equation (3.14) give

$$b_3=-a_{\alpha,\alpha}=w_{,\alpha\alpha}+\mathcal{O}(h^2). \tag{3.16}$$

In a similar manner, taking $i=\alpha$ in equation (3.9*b*) and using (3.5*b*), (3.7*b*) and (3.12), we obtain

$$b_\alpha=\mathcal{O}(h^2), \tag{3.17}$$

which, up to $\mathcal{O}(h^2)$ terms in (3.1*a*), amounts to the Kirchhoff–Love hypothesis that cross-sections remain plane during bending. Using the last of equations (3.11) with (3.17), we find $c_3=\mathcal{O}(h^2)$. Besides, taking $i=3$ gives the leading term in the incremental pressure

$$\dot{p}^{(1)}=\mathcal{A}_{33\beta\alpha}a_{\alpha,\beta}+(p+\mathcal{A}_{3333})b_3+\mathcal{O}(h^2). \tag{3.18}$$

Therefore, we have, up to $\mathcal{O}(h)$ terms,

$$a_3=0,\quad b_\alpha=0,\quad c_3=0\quad\text{and}\quad \dot{p}^{(0)}=0 \tag{3.19}$$

and the kinematics of the plate (3.1*a*) simplifies to

$$\boldsymbol{\chi}^{(1)}=w\boldsymbol{e}_3-x_3w_{,\alpha}\boldsymbol{e}_\alpha+\tfrac{1}{2}x_3^2(\nabla^2w)\boldsymbol{e}_3+\tfrac{1}{6}x_3^3\boldsymbol{c}+\cdots. \tag{3.20}$$

It is emphasized that, in the foregoing derivations, $\boldsymbol{c}$ rests undetermined.

A comparison of the results with the literature shows that the plate kinematics (3.20) encompasses eqns (3.17,23,29,30) of Kaplunov *et al.* [18]. For instance, the linear-through-the-thickness-ζ expression for the axial displacement $U^{[2]}=\ell(u_1^{(0)}+\eta^2u_1^{(2)})$, given by their eqns (3.17) and $(3.23)_1$, using equation (3.31) and (3.33) corresponds to the one-dimensional version of the second term in (3.20) here, given that $V^{[2]}=\ell(U_2^{(0,0)}+\eta^2U_2^{(0,2)}+\cdots)$ corresponds to w. Likewise, equation $(3.23)_2$ brings in the quadratic term in the transverse displacement u_2, corresponding to the third term in (3.20), which is proportional to the curvature. Finally, equation $(3.23)_3$ gives the last of equations (3.19) for the leading term in the pressure increment. Conversely, the governing equation for pre-stressed plates (7.4) cannot be directly obtained from the static limit of (3.56) of Kaplunov *et al.* [18], in the light of the fact that the latter equation is obtained under the assumption of pre-stress $\sigma_{11}=\mathcal{O}(1)$, plate thickness $2h$ and in the absence of the foundation. However, once such assumptions are modified, correspondence can be achieved. We observe that in the works of Dai & Song [27] and Wang *et al.* [28] a theory for, respectively, compressible and incompressible thin plates is developed through an expansion of the plate kinematics about the lower surface $x_3 = -h/2$, which is then fed into the governing equations.

4. Variational formulation

We begin with the following general expression for the second variation of the total potential energy of a hyperelastic body B:

$$E''_{\mathrm{B}}(0) = \int_{\mathrm{B}} \boldsymbol{\Sigma} \cdot \boldsymbol{\Gamma}\, \mathrm{d}v. \tag{4.1}$$

The integration domain B may be the configuration of any pre-stressed body and, in the following, it is identified with the set (2.1). Hereinafter, a general material and a general state of pre-stress $\sigma_{\alpha\beta}$ are considered. Besides, in order to restrict the formulation to bending, we assume that the pre-stress scales as h^2, i.e. $\sigma_{\alpha\beta} = \mathcal{O}(h^2)$. The essence of the approach is to reduce the right-hand side of (4.1) consistently to order h^3, and then obtain the reduced boundary-value problem by energy minimization [17]. Consequently, in the following derivation, terms of order higher than h^3 are neglected.

With the help of the results established in the previous sections, we can proceed to simplify the second variation (4.1). First, by substituting (3.3) and (3.6) into (4.1) and integrating along the thickness of the plate, we obtain, up to $\mathcal{O}(h^3)$ terms,

$$\begin{aligned} E''_{\mathrm{B}}(0) &= \int_{\omega} \left\{ h\Sigma^{(0)}_{ji}\Gamma^{(0)}_{ij} + \frac{h^3}{12}\left(\frac{1}{2}\Sigma^{(0)}_{ji}\Gamma^{(2)}_{ij} + \Sigma^{(1)}_{ji}\Gamma^{(1)}_{ij} + \frac{1}{2}\Sigma^{(2)}_{ji}\Gamma^{(0)}_{ij}\right)\right\} \mathrm{d}S \\ &= \int_{\omega} \left\{ h\Sigma^{(0)}_{ji}\Gamma^{(0)}_{ij} + \frac{h^3}{12}\left(\Sigma^{(0)}_{ji}\Gamma^{(2)}_{ij} + \Sigma^{(1)}_{ji}\Gamma^{(1)}_{ij}\right)\right\} \mathrm{d}S, \\ &= \int_{\omega} \left\{ h\Sigma^{(0)}_{ji}\Gamma^{(0)}_{ij} + \frac{h^3}{12}\left(\Sigma^{(0)}_{\alpha 3}b_{3,\alpha} + \Sigma^{(1)}_{\beta\alpha}a_{\alpha,\beta}\right)\right\} \mathrm{d}S, \end{aligned} \tag{4.2}$$

where, in obtaining the last expression, use has been made of the results (3.5*a*), (3.5*b*), (3.19) and of the boundary condition (3.9*a*). Indeed, the latter indicates that the term $\Sigma^{(0)}_{3\alpha}c_\alpha + \Sigma^{(0)}_{33}c_3 = \mathcal{O}(h^2)$ brings a higher order contribution which may be omitted in the round brackets. For the first term in (4.2), using equations (3.5*a*), (3.12), (3.14) and (3.19), we get

$$\Sigma^{(0)}_{ji}\Gamma^{(0)}_{ij} = \Sigma^{(0)}_{\alpha 3}w_{,\alpha} + \Sigma^{(0)}_{3\alpha}a_\alpha = (\Sigma^{(0)}_{\alpha 3} - \Sigma^{(0)}_{3\alpha})w_{,\alpha} + \mathcal{O}(h^4) = \sigma_{\alpha\beta}w_{,\alpha}w_{,\beta} + \mathcal{O}(h^4) \tag{4.3}$$

and the last expression is deduced with the aid of (2.9). It is observed that $h\Sigma^{(0)}_{ji}\Gamma^{(0)}_{ij} = \mathcal{O}(h^3)$. For the second term in (4.2), we have

$$\int_{\omega} \Sigma^{(0)}_{\alpha 3}b_{3,\alpha}\,\mathrm{d}S = \int_{\omega} \left\{ \left(\Sigma^{(0)}_{\alpha 3}b_3\right)_{,\alpha} - \Sigma^{(0)}_{\alpha 3,\alpha}b_3 \right\} \mathrm{d}S = \int_{\partial\omega} \Sigma^{(0)}_{\alpha 3}b_3 n_\alpha\,\mathrm{d}s - \int_{\omega} \left(\nabla\cdot\boldsymbol{\Sigma}^{(0)}\right)_3 b_3\,\mathrm{d}S, \tag{4.4}$$

where n_α is the unit vector normal to the mid-plane boundary $\partial\omega$. On the account of the incremental equilibrium equation in the absence of incremental body forces, i.e. div $\boldsymbol{\Sigma} = \mathbf{o}$ and with (3.9*b*), it is $\Sigma^{(0)}_{\alpha i,\alpha} = -\Sigma^{(1)}_{3i} = \mathcal{O}(h^2)$, whence, in the absence of incremental surface traction,

$$\int_{\omega} \Sigma^{(0)}_{\alpha 3}b_{3,\alpha}\,\mathrm{d}S = \mathcal{O}(h^2). \tag{4.5}$$

Thus, with equation (3.14), the second variation (4.2) becomes, to leading order,

$$E''_{\mathrm{B}}(0) = \int_{\omega} \left\{ h\sigma_{\alpha\beta}w_{,\alpha}w_{,\beta} - \frac{h^3}{12}\Sigma^{(1)}_{\beta\alpha}w_{,\alpha\beta} \right\} \mathrm{d}S. \tag{4.6}$$

Note that both terms in this expression are $\mathcal{O}(h^3)$ as we have assumed that $\sigma_{\alpha\beta} = \mathcal{O}(h^2)$. In fact, it is this very assumption that makes our expansion self-consistent.

Finally, we have, with the help of equations (3.5*b*) and (3.18) and up to $\mathcal{O}(1)$,

$$\begin{aligned}\Sigma^{(1)}_{\beta\alpha} &= \mathcal{A}_{\beta\alpha lk}\Gamma^{(1)}_{kl} + p\Gamma^{(1)}_{\beta\alpha} - \dot{p}^{(1)}\delta_{\alpha\beta}\\ &= \mathcal{A}_{\beta\alpha\gamma\delta}a_{\delta,\gamma} + \mathcal{A}_{\beta\alpha33}b_3 + pa_{\beta,\alpha} - \dot{p}^{(1)}\delta_{\alpha\beta}\\ &= (\mathcal{A}_{\beta\alpha\gamma\delta} - \mathcal{A}_{33\gamma\delta}\delta_{\alpha\beta})a_{\delta,\gamma} + (\mathcal{A}_{\beta\alpha33} - p\delta_{\alpha\beta} - \mathcal{A}_{3333}\delta_{\alpha\beta})b_3 + pa_{\beta,\alpha}\\ &= -(\mathcal{A}_{\beta\alpha\gamma\delta} - \mathcal{A}_{33\gamma\delta}\delta_{\alpha\beta})w_{,\delta\gamma} + (\mathcal{A}_{\beta\alpha33} - p\delta_{\alpha\beta} - \mathcal{A}_{3333}\delta_{\alpha\beta})w_{,\gamma\gamma} - pw_{,\alpha\beta},\end{aligned} \tag{4.7}$$

where we used (3.14) and (3.16) in the last equality. Therefore, the second variation of the potential energy can be written as

$$E''_{\mathrm{B}}(0) = \int_\omega \{h\sigma_{\alpha\beta}w_{,\alpha}w_{,\beta} + \hat{\mathcal{D}}_{\alpha\beta\delta\gamma}w_{,\delta\gamma}w_{,\alpha\beta}\}\,\mathrm{d}S, \tag{4.8}$$

whose first and second terms are quadratic forms in ∇w and $\nabla\nabla w$, and where

$$\hat{\mathcal{D}}_{\alpha\beta\delta\gamma} = \frac{h^3}{12}\{\mathcal{A}_{\beta\alpha\gamma\delta} - \mathcal{A}_{33\gamma\delta}\delta_{\beta\alpha} - \mathcal{A}_{33\beta\alpha}\delta_{\gamma\delta} + (\mathcal{A}_{3333} + p)\delta_{\alpha\beta}\delta_{\delta\gamma} + p\delta_{\beta\gamma}\delta_{\alpha\delta}\}. \tag{4.9}$$

The fourth rank tensor $\hat{\mathcal{D}}$ exhibits the major symmetry, yet it may be equally well replaced by its minor-symmetric part $\mathcal{D}$, which has six independent components out of 16. Furthermore, expanding the instantaneous moduli in powers of the thickness h about the undeformed state $\lambda_i = 1$ only the leading order term may be consistently retained, i.e. $\mathcal{D}$ is a constant tensor. Indeed, we have [29, eqn (3.16)]

$$\mathcal{A}_{\beta\alpha\gamma\delta} = \mu\delta_{\beta\gamma}\delta_{\alpha\delta} + \mathcal{O}(h) \tag{4.10}$$

and $p = \mu$, whence equation (4.9) gives

$$\mathcal{D}_{\alpha\beta\delta\gamma} = \tfrac{1}{4}\mathsf{D}\{\delta_{\beta\gamma}\delta_{\alpha\delta} + \delta_{\beta\delta}\delta_{\alpha\gamma} + 2\delta_{\alpha\beta}\delta_{\delta\gamma}\}, \tag{4.11}$$

where we have introduced the classical *plate flexural rigidity* [30]

$$\mathsf{D} = \frac{Eh^3}{12(1-\nu^2)}, \quad \nu = \frac{1}{2}, \tag{4.12}$$

in the light of the connection $\mu = E/(2(1+\nu))$ between the shear modulus, μ, and Young's modulus, E. In this context, the only non-zero elements of $\mathcal{D}$ are

$$\mathcal{D}_{1111} = \mathcal{D}_{2222} = 2\mathcal{D}_{1122} = 2\mathcal{D}_{2211} = 4\mathcal{D}_{1212} = 4\mathcal{D}_{2112} = 4\mathcal{D}_{1221} = 4\mathcal{D}_{2121} = \mathsf{D}. \tag{4.13}$$

5. Plate governing equation

Let us define the moments (per unit length) by

$$M_{\alpha\beta} = -\mathcal{D}_{\alpha\beta\delta\gamma}w_{,\delta\gamma}. \tag{5.1}$$

In particular, the *bending moments* are

$$M_{11} = -\mathsf{D}\left(w_{,11} + \tfrac{1}{2}w_{,22}\right) \quad \text{and} \quad M_{22} = -\mathsf{D}\left(w_{,22} + \tfrac{1}{2}w_{,11}\right), \tag{5.2}$$

while the *twisting moments* are

$$M_{12} = M_{21} = -\tfrac{1}{2}\mathsf{D}w_{,12}. \tag{5.3}$$

Likewise, let us define the *shearing force* (per unit length)

$$q_\alpha = h\sigma_{\alpha\beta}w_{,\beta} + M_{\beta\alpha,\beta}. \tag{5.4}$$

It follows that

$$E''_{\mathrm{B}}(0) = \int_\omega q_\alpha w_{,\alpha}\,\mathrm{d}S - \int_{\partial\omega} M_{\alpha\beta}w_{,\alpha}n_\beta\,\mathrm{d}s. \tag{5.5}$$

Taking the first variation of (5.5) with respect to w, we obtain

$$\frac{1}{2}\delta E_{\mathrm{B}}''(0) = -\int_{\omega} q_{\alpha,\alpha}\delta w\,\mathrm{d}S + \int_{\partial\omega}(q_\alpha \delta w - M_{\beta\alpha}\delta w_{,\beta})n_\alpha\,\mathrm{d}s. \tag{5.6}$$

The last term in the boundary integral is better expressed in terms of the normal and the tangential (with respect to the boundary) derivatives, making use of the formula [31, §36.4]

$$\delta w_{,\beta} = \tau_\beta \frac{\partial \delta w}{\partial \tau} + n_\beta \frac{\partial \delta w}{\partial n}. \tag{5.7}$$

Setting to zero the first variation inside ω gives the *plate governing equation*

$$\nabla \cdot \boldsymbol{q} = 0 \quad \text{in } \omega, \tag{5.8}$$

which reads

$$\mathsf{D}\nabla^2 w - h\sigma_{11}w_{,11} - 2h\sigma_{12}w_{,12} - h\sigma_{22}w_{,22} = 0 \tag{5.9}$$

and it amounts to enforcing out-of-plane equilibrium of a plate element. Here, $\nabla^2 w = w_{,1111} + 2w_{,1122} + w_{,2222}$ is the biharmonic operator applied to w. In the same fashion, setting to zero the first variation of the boundary integral yields the *natural boundary conditions*

$$V_n \delta w = 0 \quad \text{and} \quad M_n \delta\frac{\partial w}{\partial n} = 0, \quad \text{on } \partial\omega, \tag{5.10}$$

where $V_n = q_\alpha n_\alpha + (\partial/\partial\tau)(\tau_\beta M_{\beta\alpha} n_\alpha)$ is the well-known *Kirchhoff equivalent shearing force* and $M_n = n_\beta M_{\beta\alpha} n_\alpha$ is the *bending moment*, both acting on a surface with unit normal n_α. Setting $\boldsymbol{n} = \cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2$, they read

$$V_n = q_n - M_{n\tau,1}\sin\theta + M_{n\tau,2}\cos\theta \tag{5.11}$$

and

$$M_n = \tfrac{1}{2}(M_{11} + M_{22}) + \tfrac{1}{2}(M_{11} - M_{22})\cos 2\theta + M_{12}\sin 2\theta, \tag{5.12}$$

where we have let $q_n = q_1\cos\theta + q_2 \sin\theta$ and

$$M_{n\tau} = \tfrac{1}{2}(M_{22} - M_{11})\sin 2\theta + M_{12}\cos 2\theta. \tag{5.13}$$

6. Stress distribution and comparison with the classical theory

Equations (3.15), (4.7) and (4.9) give the normal stress distribution (no sum over α in this section)

$$\Sigma_{\alpha\alpha} = x_3 \Sigma_{\alpha\alpha}^{(1)} + \mathcal{O}(h^2) = -\frac{12}{h^3} x_3 \mathcal{D}_{\alpha\alpha\beta\gamma} w_{,\beta\gamma} + \mathcal{O}(h^2), \tag{6.1}$$

that is, using (4.13) and to leading order,

$$\Sigma_{11} = -\frac{12}{h^3}x_3\mathsf{D}(w_{,11} + 2w_{,22}) \quad \text{and} \quad \Sigma_{22} = -\frac{12}{h^3}x_3\mathsf{D}(w_{,22} + 2w_{,11}). \tag{6.2}$$

It is a straightforward matter to show that, up to $\mathcal{O}(h^3)$, the bending moments (5.2) are obtained integrating along the plate thickness the normal stress couples, i.e.

$$M_{\alpha\alpha} = \int_{-h/2}^{h/2} x_3 \Sigma_{\alpha\alpha}\,\mathrm{d}x_3. \tag{6.3}$$

Similarly, for the in-plane shear stress, we have, to leading order,

$$\Sigma_{\alpha\beta} = x_3 \Sigma_{\alpha\beta}^{(1)} = -\frac{12}{h^3} x_3 \mathcal{D}_{\alpha\beta\gamma\delta} w_{,\delta\gamma}, \tag{6.4}$$

which is symmetric owing to the minor symmetry of $\mathcal{D}_{\alpha\beta\gamma\delta}$ and it reads

$$\Sigma_{12} = \Sigma_{21} = -\frac{6}{h^3}x_3\mathsf{D}w_{,12}. \tag{6.5}$$

By integrating along the thickness and up to $\mathcal{O}(h^3)$, it gives the twisting moment (5.3)

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} x_3 \Sigma_{\alpha\beta}\, \mathrm{d}x_3. \tag{6.6}$$

Equations (3.9) allow us to write the out-of-plane stress $\Sigma_{3\alpha}$ consistently at $\mathcal{O}(h^2)$

$$\Sigma_{3\alpha} = \frac{1}{2}\left(x_3^2 - \frac{h^2}{4}\right)\Sigma_{3\alpha}^{(2)}, \tag{6.7}$$

which amounts to the classical ad hoc assumptions that the out-of-plane stress is of higher order than the in-plane stress and its distribution along the thickness is parabolic (cf. [30]). From equations (3.5*c*) and (2.8) and to leading order,

$$\Sigma_{3\alpha}^{(2)} = \mathcal{A}_{3\alpha\beta 3}(c_\beta + w_{,\gamma\gamma\beta}) + p(c_\alpha + w_{,\gamma\gamma\alpha}), \tag{6.8}$$

which, in the light of (4.10), reduces to

$$\Sigma_{3\alpha}^{(2)} = \mu(c_\alpha + w_{,\gamma\gamma\alpha}). \tag{6.9}$$

The so-far-undetermined vector c_α may be obtained, to leading order, through the incremental equilibrium equation div $\boldsymbol{\Sigma} = \mathbf{o}$, i.e.

$$\Sigma_{3\alpha}^{(2)} = -\Sigma_{\beta\alpha,\beta}^{(1)}, \tag{6.10}$$

whence its components are

$$c_1 = 3(w_{,111} + 3w_{,122}) \quad \text{and} \quad c_2 = 3(w_{,222} + 3w_{,112}). \tag{6.11}$$

Thus, through equations (5.4), (6.4), (6.6) and (6.10), the shearing force may be related to the out-of-plane stress distribution and the pre-stress as

$$\begin{aligned} q_\alpha &= h\sigma_{\alpha\beta} w_{,\beta} + \int_{-h/2}^{h/2} x_3 \Sigma_{\beta\alpha,\beta}\, \mathrm{d}x_3 = h\sigma_{\alpha\beta} w_{,\beta} + \int_{-h/2}^{h/2} x_3^2 \Sigma_{\beta\alpha,\beta}^{(1)}\, \mathrm{d}x_3 \\ &= h\sigma_{\alpha\beta} w_{,\beta} - \int_{-h/2}^{h/2} x_3^2 \Sigma_{3\alpha}^{(2)}\, \mathrm{d}x_3 = h\sigma_{\alpha\beta} w_{,\beta} - \frac{h^3}{12}\Sigma_{3\alpha}^{(2)}, \end{aligned} \tag{6.12}$$

which gives the classical formula [30, §21], corrected to account for the pre-stress, namely

$$\Sigma_{3\alpha\,\max} = \frac{3}{2h}(q_\alpha - h\sigma_{\alpha\beta} w_{,\beta}) = \mathcal{O}(h^2). \tag{6.13}$$

Besides, integrating equation (6.7) along the thickness and comparing with the last of equations (6.12), we may write

$$q_\alpha = h\sigma_{\alpha\beta} w_{,\beta} + \int_{-h/2}^{h/2} \Sigma_{3\alpha}\, \mathrm{d}x_3, \tag{6.14}$$

which corresponds to the classical definitions of Q_x and Q_y, respectively, in eqns (106,107) of [30], corrected to incorporate the pre-stress. The bending moments (5.2) as well as the twisting moment (5.3) correspond to the classical definitions (101,102) of Timoshenko & Woinowsky-Krieger [30, §21]. The Kirchhoff shearing force (5.11) amounts to the corresponding definition (g) of [30], provided that we take $M_{12} = M_{yx} = -M_{xy}$. The governing equation (5.9) coincides with the classical equation for combined bending and compression (or tension) of thin plates, first derived by Saint Venant [32]. Finally, we observe that equation (3.20), up to $\mathcal{O}(h)$-terms, parallels the classical plate kinematics, the remaining terms being a higher order correction.

7. Edge wrinkling solution

Let us consider the hyperelastic plate B to occupy the semi-infinite region (figure 1)

$$\mathrm{B} = \left\{ |x_1| < \infty,\ 0 \le x_2 < \infty,\ |x_3| \le \frac{h}{2} \right\}. \tag{7.1}$$

From now on, we assume that the $\boldsymbol{e}_i$ are directed along the principal axes of the underlying homogeneous deformation, in which case the deformation gradient has a diagonal representation, namely

$$\boldsymbol{F} = \mathrm{grad}\,\boldsymbol{\chi}^{(0)} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3). \tag{7.2}$$

In particular, given that the plate edge corresponds to the plane $x_2 = 0$, we have $\theta = \pi/2$ and the boundary conditions (5.11) and (5.12) now specialize to

$$2M_{21,1} + M_{22,2} = 0 \quad \text{and} \quad M_{22} = 0, \tag{7.3}$$

respectively. The governing equation equation (5.9) is now supplemented by the Winkler term (2.12)

$$\mathsf{D}\nabla^4 w - h\sigma_{11} w_{,11} - h\sigma_{22} w_{,22} + \kappa w = 0. \tag{7.4}$$

It is emphasized that, for the expression (2.11) of the second variation $E''(0)$ to be consistent, we need to assume that $\kappa_1 = \kappa/h^3$ is of order $\mathcal{O}(1)$. This assumption is analogous to the scaling requirement for a thin-coated half-plane [33]. Similarly, the boundary conditions (7.3) may be written as

$$w_{,11} + 2w_{,22} = 0 \quad \text{and} \quad 3w_{,112} + 2w_{,222} = 0. \tag{7.5}$$

We look for a *wrinkling solution* to equation (7.4), which varies sinusoidally along the edge and decays away from it as $x_2 \to \infty$, in the form [9,13]

$$w(x_1, x_2) = \Re\left\{ [A_1 \exp(-\gamma_1 k x_2) + A_2 \exp(-\gamma_2 k x_2)] \exp(\imath k x_1) \right\}, \tag{7.6}$$

where A_1, A_2 are yet-undetermined amplitude constants, γ_1, γ_2 the attenuation coefficients, such that $\Re(\gamma_\alpha) > 0$ and $k > 0$ is the wavenumber (see figure 2 for an illustration). Substitution into the plate equation (7.4) shows that γ_1 and γ_2 are the roots of the bi-quadratic equation

$$\gamma^4 - 2\gamma^2 + 1 + \frac{\sigma + \hat{\kappa}}{d_0} = 0, \tag{7.7}$$

where the following $\mathcal{O}(1)$ quantities (with physical dimension of stress) have been introduced: $d_0 = \mathsf{D}/h^3 = \mu/2$, $\sigma = (\sigma_{11} + \sigma_{22})/(kh)^2$ and $\hat{\kappa} = \kappa_1/k^4 = \kappa h/(kh)^4$.

Enforcing the boundary conditions (7.5) gives a homogeneous linear algebraic system of two equations in the two unknowns A_1 and A_2

$$\begin{aligned} (1 - 2\gamma_1^2)A_1 + (1 - 2\gamma_2^2)A_2 = 0 \quad \text{and} \\ [3 - 2\gamma_1^2]\gamma_1 A_1 + [3 - 2\gamma_2^2]\gamma_2 A_2 = 0, \end{aligned} \tag{7.8}$$

which admits non-trivial solutions provided that the determinant of the coefficient matrix is zero. Hence we arrive at the *bifurcation condition*,

$$(\gamma_1 - \gamma_2)[4\gamma_2\gamma_1 + 3 - 2(\gamma_1^2 + \gamma_2^2) + 4\gamma_1^2\gamma_2^2] = 0. \tag{7.9}$$

The bifurcation condition (7.9) is satisfied whenever $\gamma_1 = \gamma_2$ or when the term in square brackets vanishes. It can be shown that the former case is spurious, whereas the latter yields

$$\sqrt{1 + \frac{\sigma + \hat{\kappa}}{d_0}} + \frac{\sigma + \hat{\kappa}}{d_0} + \frac{3}{4} = 0 \tag{7.10}$$

and the plus sign has been chosen for the square root in the light of the requirement $\Re(\gamma_\alpha) > 0$. Equation (7.10) expresses the *bifurcation criterion* for the appearance of wrinkles on the edge of

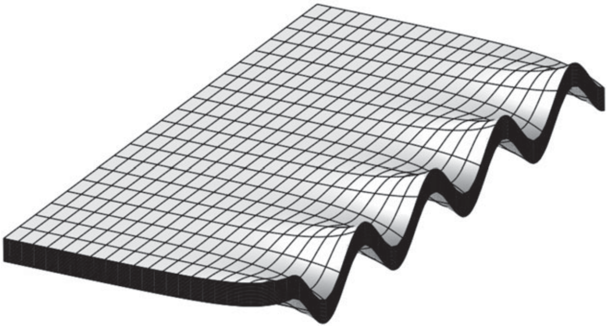

Figure 2. Buckling wrinkles with wavenumber k localized near the edge of a compressed thin plate with thickness h.

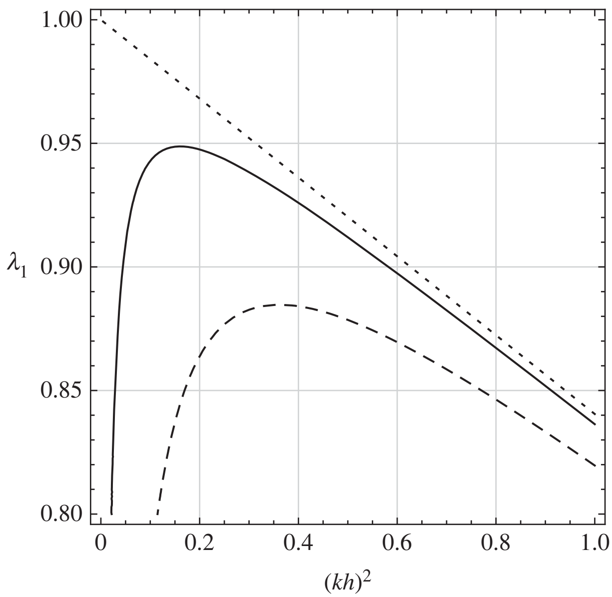

Figure 3. Bifurcation curves for the appearance of edge wrinkles with scaled wavenumber kh on a thin plate made of incompressible elastic material with flexural rigidity D, resting on a Winkler foundation with elastic modulus $\kappa = 3\ell_s^{-4}\mathsf{D}$. The curves are for uni-axial pre-stress and foundation relative compliance $h/\ell_s = 0$ (dotted), 1/3 (solid) and 1/2 (dashed). In the first case (no Winkler foundation), the plate buckles as soon as it is compressed laterally.

a semi-infinite plate compressed by a lateral stress. It can be rationalized by squaring and then solved to yield

$$\sigma_{11} + \sigma_{22} = -(kh)^2 \left[\tfrac{1}{4} d_0 (1 + 2\sqrt{2}) + \hat{\kappa} \right], \tag{7.11}$$

which shows that, as it was assumed, $\sigma_{11} + \sigma_{22} = \mathcal{O}(h^2)$. Besides, in the absence of the supporting foundation (i.e. $\kappa_1 = 0$), equation (7.11) is trivially satisfied by $kh = 0$, which amounts to a zero pre-stress condition and, consequently, to a critical buckling stretch of $\lambda = 1$. In other words, without the substrate, the plate would buckle as soon as it is laterally compressed.

Substituting the expansion $\lambda_\alpha = 1 + (kh)^2 \lambda_\alpha^{(2)}$ and retaining terms up to h^2, we find the bifurcation condition

$$\frac{1}{2}(\lambda_1 + \lambda_2) = 1 - \frac{(kh)^2}{96} \left(1 + 2\sqrt{2} + \frac{8}{(\ell_s k)^4} \right) \tag{7.12}$$

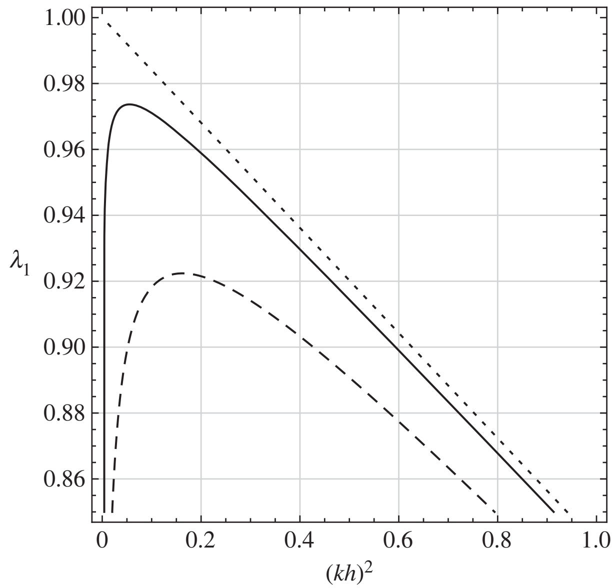

Figure 4. Bifurcation curves for the appearance of edge wrinkles on a thin plate resting on a local foundation whose elastic modulus $\kappa = (h/\ell_s)^4 \mu k$ is proportional to the wavenumber k. The curves are for uni-axial pre-stress and foundation relative compliance $h/\ell_s = 0$ (dotted), 1/3 (solid) and 1/2 (dashed).

in terms of the *characteristic length* $\ell_s = \sqrt[4]{\mu/\kappa_1} = \sqrt[4]{3\mathsf{D}/\kappa}$, expressing the relative flexural rigidity of the plate compared with the stiffness of the foundation. It is observed that $\ell_s = \mathcal{O}(1)$. In the case of uni-axial pre-stress, $\lambda_2 = 1/\sqrt{\lambda_1}$ and equation (7.12) becomes

$$\lambda_1 = 1 - \frac{(kh)^2}{6}\left(\frac{1}{4} + \frac{\sqrt{2}}{2} + \frac{2}{(\ell_s k)^4}\right). \tag{7.13}$$

In figure 3, the bifurcation curves (7.13) are plotted considering three values of the foundation relative compliance h/ℓ_s. Typically, curves go through a maximum which determines the effective critical stretch of contraction as well as the edge wrinkles wavenumber. A similar pattern is shown in figure 4 where the foundation stiffness is taken to be proportional to the wavenumber k, which is the situation of a linear elastic half-space foundation considered in [20,34]. By contrast, when the foundation stiffness scales as the wavenumber squared, bifurcation curves become straight lines and edge wrinkling starts at zero wavenumber. This is indeed the nonlinear correction considered by Brau *et al.* [34] and it may be argued that appearance of edge wrinkles at zero wavenumber is a good test for such an assumption. Furthermore, any dependence of the foundation stiffness on powers of the wavenumber greater than 2 leads to bifurcation at $\lambda_1 = 1$ and zero wavenumber.

8. Body versus edge wrinkling

The body wrinkling solution takes the form

$$w(x_1, x_2) = \Re\{\exp[\imath k(n_1 x_1 + n_2 x_2)]\}, \quad n_1^2 + n_2^2 = 1, \tag{8.1}$$

which, when plugged into equation (7.4) and expanded for $\lambda_\alpha = 1 + (kh)^2 \lambda_\alpha^{(2)}$, gives the wrinkling condition

$$(3 - 2\lambda_1 - \lambda_2)n_1^2 + (3 - \lambda_1 - 2\lambda_2)n_2^2 = \frac{1}{4}(kh)^2\left[1 + 2\frac{1}{(\ell_s k)^4}\right]. \tag{8.2}$$

This equation may be plotted for different values of n_2, thus giving the bifurcation curves for bulk wrinkling. In the special case of uni-axial pre-stress along x_1 (i.e. $n_2 = 0$, $n_1 = 1$), equation

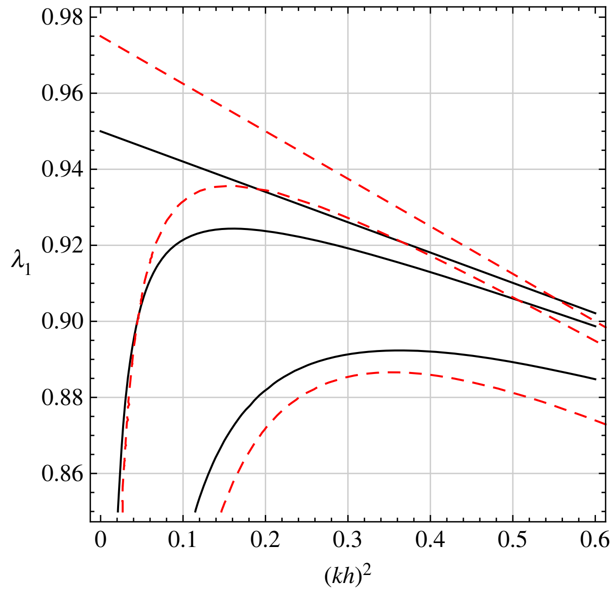

Figure 5. Edge (solid) versus body (dashed) bifurcation curves for a thin incompressible elastic plate resting on a Winkler foundation with relative compliance $h/\ell_s = 0, 1/3, 1/2$ in a state of bi-axial pre-stretch ($\lambda_1 < 1$, $\lambda_2 = 1.05$). Body wrinkling takes place in compression and parallel to the edge (i.e. $n_2 = 0$); it occurs prior to edge wrinkling for $h/\ell_s = 0$ and $1/3$ and after it for $h/\ell_s = 1/2$. (Online version in colour.)

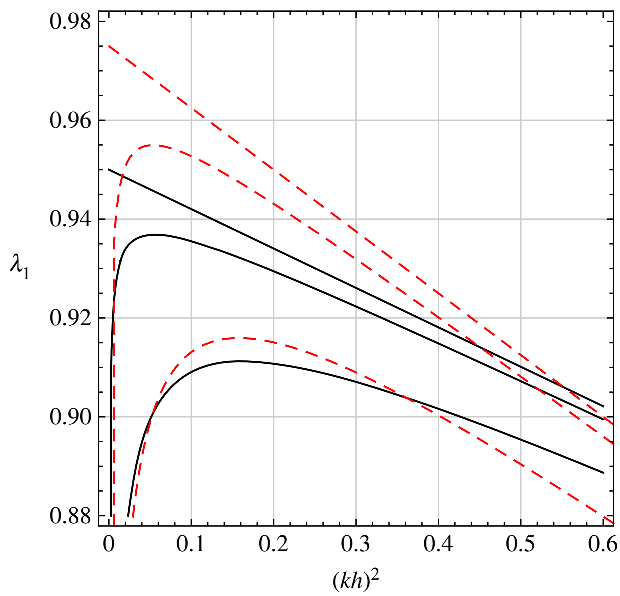

Figure 6. Edge (solid) versus body (dashed) bifurcation curves for a thin incompressible elastic plate resting on a local foundation with stiffness $\kappa = (h/\ell_s)^4 \mu k$ proportional to the wavenumber k and relative compliance $h/\ell_s = 0, 1/3, 1/2$, in a state of bi-axial pre-stretch ($\lambda_1 < 1$, $\lambda_2 = 1.05$). Body wrinkling takes place in compression and always prior to edge wrinkling. (Online version in colour.)

(8.2) simplifies to

$$\lambda_1 = 1 - \frac{(kh)^2}{6}\left[1 + \frac{2}{(\ell_s k)^4}\right]. \tag{8.3}$$

Comparing this equation with (7.13), it is concluded that, under uni-axial pre-stress, body wrinkling takes place in compression with $n_2 = 0$ at a critical stretch $\lambda^*_{1\,\mathrm{body}}$ which is a little smaller than the corresponding threshold for edge wrinkling $\lambda^*_{1\,\mathrm{edge}} > \lambda^*_{1\,\mathrm{body}}$, i.e. edge wrinkling is preferred to body wrinkling. This result holds regardless of the foundation stiffness and it is not affected by the foundation responding to the wrinkles' wavenumber. However, in the general bi-axial pre-stressed case, this is not always the case. Indeed, for $n_2 = 0$, equation (8.2) gives

$$\lambda_1 = \frac{3-\lambda_2}{2} - \frac{(kh)^2}{8}\left[1 + \frac{2}{(\ell_s k)^4}\right] \tag{8.4}$$

which may be greater than the edge wrinkling bifurcation curve for $\lambda_2 > 1$. Figure 5 shows that body buckling may be preferred to edge buckling in a bi-axially pre-stretched scenario, where transverse extension $\lambda_2 = 1.05$ favours body wrinkle formation in compression, i.e. $\lambda_1 < 1$, up to moderate values of foundation compliance. Similarly, figure 6 compares body and edge wrinkling for a thin plate supported by a Winkler foundation whose stiffness is proportional to the wavenumber k: in this situation, body wrinkling is preferred to edge wrinkling up to large values of foundation compliance.

9. Conclusion

In this paper, a consistent model for flexural edge wrinkling of bi-axially pre-stressed thin incompressible elastic plates, supported by a local elastic foundation, is developed and solved. The governing equations and boundary conditions are derived from minimizing a reduced form for the second variation of the potential energy, which is obtained by expanding the three-dimensional kinematics through the plate thickness. Small deviations from the homogeneously pre-stressed state are investigated. Consistency of the expansion demands that, to obtain purely flexural deformations, the pre-stress scales as the plate thickness squared. Besides, it demands that the foundation stiffness scales as the plate thickness cubed. Within such assumptions, the classical Kirchhoff–Love theory of pre-stressed elastic plates is obtained. Furthermore, the ad hoc assumptions on the stress distribution are also retrieved, while parabolic and cubic terms are introduced to correct the classical linear (along the thickness) plate kinematics. Edge wrinkling is described by means of bifurcation curves and it is compared with body wrinkling. It is found that edge wrinkling always occurs prior to body wrinkling in a uni-axially pre-stressed situation, regardless of the foundation stiffness. The bifurcation landscape is more involved in a bi-axial condition and body wrinkling may precede edge wrinkling for moderate foundation stiffness. The situation where the foundation reaction depends on the wavenumber k is also discussed. In particular, it is observed that dependence on k^2 determines buckling at zero wavenumber while dependence on k^β, $\beta > 2$ produces buckling at $\lambda = 1$. Such results may be employed to infer the mechanical behaviour of the supporting matrix in flexible embedded systems, with special regard to biological tissues or organic thin-films.

Authors' contributions. M.D. and A.N. developed the model and the incremental approach; Y.F. carried out the variational reduction. All authors gave final approval for publication.
Competing interests. We declare we have no competing interests.
Funding. M.D. gratefully acknowledges partial funding from the Università degli Studi di Modena e Reggio Emilia. A.N. is grateful to the National Group of Mathematical Physics (GNFM-INdAM) for partial support through the 'Progetto Giovani Ricercatori 2015' scheme, prot. U2015/000125.

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