PHILOSOPHICAL TRANSACTIONS A

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Introduction

Cite this article: Destrade M, Murphy J, Saccomandi G. 2019 Rivlin's legacy in continuum mechanics and applied mathematics. *Phil. Trans. R. Soc. A* **377**: 20190090.
http://dx.doi.org/10.1098/rsta.2019.0090

Accepted: 05 February 2019

One contribution of 12 to a theme issue 'Rivlin's legacy in continuum mechanics and applied mathematics'.

Subject Areas:
applied mathematics, mechanics

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Rivlin's legacy in continuum mechanics and applied mathematics

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1. Introduction

Over a long and distinguished career, Ronald Rivlin (figure 1) published more than 200 scientific papers. He was a highly innovative scientist who made seminal contributions in all areas of continuum mechanics. He was one of the last *savants*, equally proficient in solid and fluid mechanics and in the mathematical methods needed to advance these disciplines. Although it was characteristic of scientists at the time of Cauchy and Navier, or even Poincaré, this spread of knowledge no longer seems possible due to the rapid expansion of research that has occurred over the last 50 years.

Rivlin was interested in developing not only theories of material modelling but also in their application to real-world problems. Rivlin's ideas have therefore attracted the interest of physicists, material scientists, engineers and applied mathematicians. His list of co-authors includes Millard Beatty (University of Nebraska), Jerald Ericksen (Johns Hopkins University), Alan Gent (University of Akron), Albert Green (Oxford University), Leonard Mullins (British Rubber Producers' Research Association), Anthony Spencer (University of Nottingham), Alan Thomas (University College London), Richard Toupin (IBM Thomas J. Watson Research Center). He was the PhD supervisor of, among others, Michael Carroll (Rice University), Michael Hayes (University College Dublin) and Allen Pipkin (Brown University). Together with Rivlin, the people listed here,

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royalsocietypublishing.org/journal/rsta *Phil. Trans. R. Soc. A* **377**: 20190090

Figure 1. Ronald S. Rivlin [1915–2005]. (Online version in colour.)

along with their students and collaborators, were at the forefront of the development and application of continuum mechanics.

Continuum mechanics is a fundamental science at the nexus of mechanics, engineering science and applied mechanics. Continuum mechanics flourished in the aftermath of the Second World War, with its axiomatic approach and general methodology attracting great attention in the then rapidly expanding universities. Inevitably, after this initial surge of interest, a waning followed due to the next generation of scientists being forced to concentrate on narrower areas of expertise because of the sheer volume of global research output. However, over the past 20 years, continuum mechanics has experienced a renaissance, with renewed interest in the field generated by the need to model *nonlinear acoustics, soft matter, biological soft tissues and active materials*. These complex systems require methodologies that are at the core of continuum mechanics, with the result that many scientists from different backgrounds are embracing continuum mechanics once again and the field now has a wide and substantial audience.

Many of the fundamental problems of continuum mechanics have been formulated and substantially advanced by Rivlin. For example, the current analysis of stability problems in the biomechanics of growth relies on the theory of the superposition of small deformations on finite deformations initially developed by Rivlin, Hayes and others. The current interest in modelling residual stresses relies on the theory of invariants developed by Rivlin and co-workers and the analysis of active materials is based on the Rivlin and Carroll's formulation. This is just to cite a few of the recent applications of Rivlin's theories.

The aims of this theme issue are to broaden the appeal of continuum mechanics and to attract those scientists who appreciate the power of its problem-solving methodologies but whose background hinders their immediate appreciation of the technical details.

There is also a historical reason to the timeliness of this thematic issue. Many of the students of Rivlin have passed away and it is time to collect the remaining knowledge of the community that was in direct contact with him. Moreover, it was in *Philosophical Transactions* that Rivlin laid down the mathematics and physics of what is now called Nonlinear Elasticity, with an impressive series of 10 articles published by this journal from 1948 to 1955 (figure 2).

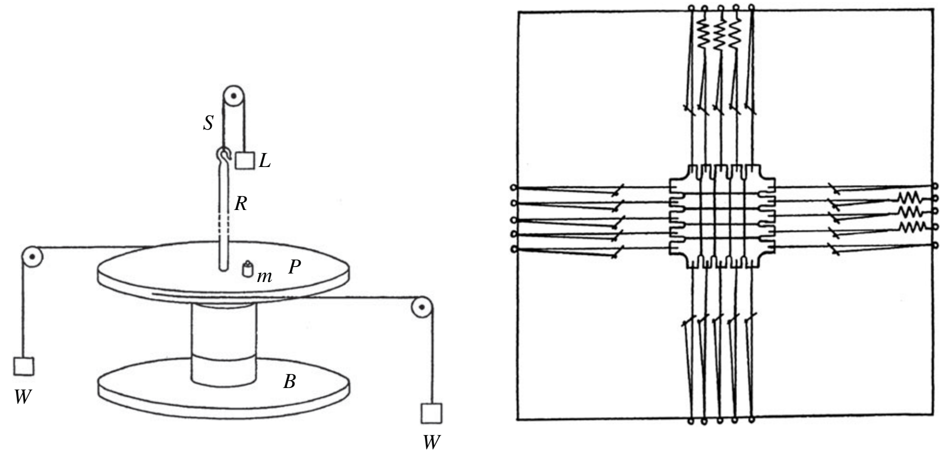

Figure 2. Rivlin was a very thorough and astute experimentalist. In one of his 10 articles published from 1948 to 1955 in *Philosophical Transactions*, he proposed protocols to measure the moment and normal force required to twist a rubber cylinder, and to measure stresses and strains in the bi-axial stretching of a rubber membrane [1].

Rivlin's collected papers were published in 1997 as a two-volume set [2] and classified under the following topics:

— Isotropic finite elasticity*,
— Anisotropic finite elasticity*, Kinematic constraints,
— Superposition of small deformations on finite deformations in elastic materials*, Stability*,
— Constitutive equations, Invariants,
— Internal variables theories,
— Non-Newtonian fluids*,
— Electromagnetism*,
— Fracture,
— Waves in viscoelastic materials*,
— Crystal physics,

plus some miscellaneous and general papers.

This issue presents a collection of research and review articles covering the starred topics above from a modern perspective, and exposes how vibrant Rivlin's legacy is.

2. Historical context

Rivlin operated in a special period in the history of science, which can be considered as a sort of Renaissance of Natural Philosophy.

After the Second World War, a group of talented mechanicians and mathematicians worked hard at transforming continuum mechanics into a *Rational Mechanics* programme. Ronald Rivlin and Clifford Truesdell were undoubtedly the leaders of this programme. In the early days, Rivlin and Truesdell had cordial relations based on mutual respect, but later on their relationship deteriorated, as some of Truesdell's followers were encouraged to develop mechanics as a pure axiomatic subject in the spirit of David Hilbert. This was a departure from the spirit of *rational mechanics* as pursued by Paul Appel in France or Tullio Levi–Civita in Italy, where the challenge was to combine deep theoretical analysis with concrete practicality. If only this clash of characters and schools between Rivlin and Truesdell had been avoided, then continuum mechanics would have enjoyed an unassailable advantage in researchers and resources.

For a long time, it was unclear whether continuum mechanics was a discipline of interest to applied mathematicians only, or to engineers, or to physicists. It is easy to see why engineers are interested in the applications of fluid and solid mechanics, and why applied mathematicians would be attracted to concepts and tools closely associated with continuum mechanics, such as scaling, dimensional analysis, stability theory, asymptotic analysis, Fourier methods, homogenization and finite-element methods. However, to date, the subject of continuum mechanics is almost absent from the typical undergraduate curriculum in physics. Things are changing quickly now, as physicists, biologists, material scientists and biomedical researchers are entering the areas spanned by the subject, with an explosion of research activity in the modelling, understanding and applications of flowing fluids, atmospheric circulation, active materials and the behaviour of soft materials such as dielectrics, membranes, polymeric gels, emulsions, metamaterials and biological materials.

3. Isotropic finite elasticity

The achievements of Rivlin in formulating the theory of isotropic finite elasticity are unparalleled, so much so that the fundamental model of incompressible nonlinear elasticity, the *Mooney–Rivlin strain-energy function*, is named after him. It reads

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \tag{3.1}$$

where C_1, C_2 are positive constants and I_1, I_2 are the first two principal invariants of strain.

The solutions of the canonical boundary value problems obtained by Rivlin form the basis of many of the developments of the theory (e.g. [3]). Today, many branches of physics and engineering science use the theory of nonlinear elasticity initiated by Rivlin. A striking current application of this theory is its ability to include anisotropy (§4) and study the mechanical response of soft tissue and the modelling of the complex interactions between elastic response, anisotropy and residual stresses [4,5].

More than 70 years after Rivlin initiated the topic, there is still huge research activity and effort devoted to the quest of *the* strain-energy for natural rubber. There has been and still is a continuous flow of new models for rubber coming out in the literature, to be confronted by experimental data, which is puzzling because a mathematical model must be robust and general, and cannot rely on a single specific functional form only.

The starting point of the Mooney–Rivlin model can be found in the original paper by Mooney [6]:

> When a sample of soft rubber is stretched by an imposed tension, neither the force-elongation nor the stress-elongation relationship agrees with Hooke's law. But, if the sample is sheared by a shearing stress, or traction, Hooke's law is obeyed over a very wide range in deformation. . . . Another simplifying feature in the elasticity of rubber lies in the fact that deformations are normally produced without any appreciable change in volume.

Rivlin was able to substantiate this point of view by clarifying several theoretical and experimental aspects of rubber mechanics. He produced the mathematical model (3.1), which 'led to the exploration of the nonlinear theory of elasticity in deep and unexpected ways, yielded significative classes of non-homogeneous exact solutions and provided a new perspective to the interpretation of experimental data' [7]. Its applicability holds quite widely, even though the Mooney–Rivlin model is special in several respects. For instance, it is only an example among a huge family of models that possesses the requirements given by Mooney [8]; its mathematical structure leads to predictions that are too restrictive (finite-amplitude transverse waves should obey linear propagation laws); and the comparison with experiments is good only for a restricted range of deformation.

In any case, the seed was sowed. The highly successful Ogden strain-energy density function [9] came later and although it seemed to start from a completely different perspective, it was clearly grounded in Rivlin's work. Rivlin's theory expresses the strain energy function of an isotropic solid in terms of the first three principal invariants of strain, whereas Ogden uses a different set of invariants. The difference in choosing Rivlin's set of invariants as opposed to another set caused a lot of debate but an artificial controversy, because in a finite-dimensional space, any set of invariants is related to any other by a bijective map; the only difference generated is akin to a numerical pre-conditioning of the data (in fact, Rivlin was so concerned with this contrast that the last two papers he ever wrote in his late eighties were about the comparison between the Valanis–Landel approach and his own approach [10,11]). The Ogden model was the concrete realization of the power and potential of the *general theory of large deformation isotropic elasticity* developed by Rivlin because it finally became possible to fit accurately theoretical stress–strain curves to experimental data for a variety of deformations and a large range of strains.

Today we know much more about this general theory: we have a clear idea about the numerical turning points in the fitting of the phenomenology theory with experimental data [12]; we know that the theory is really robust and the actual functional forms do not really matter [7]; we know how to relate the phenomenological point of view to the mesoscopic description of polymeric networks [3]; and so on.

In the present volume, we publish a posthumous paper[1] by Michael Carroll (a student of Rivlin's), which precisely proposes a refinement of some ideas connecting a phenomenological theory to mesoscopic quantities and structures of physical interest. The paper tackles a problem that was approached incompletely during Rivlin's age. It is well known that there are two different approaches to studying rubber elasticity: the statistical or kinetic theory approach, and the phenomenological approach based on the theory of continua. Because the statistical theory deals with fundamental physical quantities and the phenomenological theory deals with mathematical assumptions, there is a common belief that the former is somehow superior to the latter. However, we must remember that the statistical mechanics of an amorphous material is not a completely rigorous discipline, as it relies on a large amount of *ad hoc* assumptions. The main discrepancy between the two approaches lies in the role and importance of the I_2 term in the Mooney–Rivlin model (3.1). Indeed, the standard methods of statistical mechanics cannot yield a macroscopic model of a polymeric network with a dependence on the second invariant of the strain. But this dependence is crucial to model the real-world behaviour of soft matter, and indeed some universal relations have established general and quantitative measures of the importance and the need of the I_2 dependence [13] (for example, a material with a strain energy depending on I_1 only cannot exhibit the Poynting effect in simple shear [14].) Trying to reconcile the two attempts was a problem for a long time (see, for example, [15] for an evocative title on this subject), and eventually more sophisticated models of statistical mechanics managed to introduce this dependence in the derivation of the macroscopic model of rubber.

The paper by Carroll was written 10 years ago and we must put its significance in an updated context. First of all, we point out that the idea used by Carroll of connecting the phenomenological and the statistical theories is due to Beatty [16]. Next, we note that there is a large literature devoted to approximating, as Carroll does, the inverse Langevin function in the context of rubber elasticity. This function was introduced by Arruda & Boyce [17] to derive their eight-chain model. The question of a proper approximation for the inverse Langevin function can be bypassed by an approach à la Gent [18], which provides a very good approximation of this special function in several circumstances [19]. Finally, Destrade *et al*. [7] recently proposed a detailed discussion and an updating of the present paper by Carroll [20].

[1]This previously unpublished paper was sent to two of us (M. Destrade and G. Saccomandi) in 2010. It was intended for a Special Issue of the *International Journal of Engineering Science* dedicated to K.R. Rajagopal (volume 48, Issue 11, 2010). Unfortunately, it was not possible to process the paper then as the source file was corrupted and some plots were missing. We asked for a definite version but Mike Carroll, who was in ill health, never replied. Jim Casey (UC Berkeley) also tried to source the original files but was ultimately unsuccessful. For this issue, the three of us pieced together the two versions of the paper that we had and we reproduced 'as is' the various corrupted plots. The paper was sent by *Philosophical Transactions A* to two independent anonymous referees and we took the initiative to implement ourselves the minor revisions they recommended.

4. Anisotropic finite elasticity

The mark left by Rivlin in nonlinear elasticity is even more significant when we move from rubber mechanics to biomechanics, with applications in the mechanics of biological soft tissues and biomaterials [21], of swelling [22] and of morpho-elasticity [23].

Biomaterials are often reinforced by families of parallel fibres (made for example of collagen), which calls for the introduction of anisotropy into the modelling. Here, Rivlin played a fundamental role in developing and analysing the theory of soft anisotropic materials. He was able to treat this problem from the perspective of the continuum, by introducing anisotropic invariants in addition to the principal invariants of strain. Current models of fibre-reinforced materials have now reached a high degree of sophistication in many applications, such as the modelling of arteries [24] or skin [25], and can explain many *exotic* material behaviours, such as the reverse Poynting effect [26], observed in some fibre-reinforced hydrogels [27].

Ben Amar *et al*. [28] provide a compelling case for the introduction of finite elasticity models into the description and understanding of many biological processes. In their review article, they cover morphogenesis, tissue mechanics, tissue growth and active cell mechanics. They go on to explain that there is a great need to work at scales intermediate between the microscopic (cells) and the macroscopic (tissues).

Growth theory is closely related to inelasticity and plasticity theory, as noted and exploited by Rodriguez *et al*. [29]. In an opinion piece here, Miles Rubin [30] proposes an Eulerian formulation of inelasticity which can be used to model viscoelastic fluids and growth of anisotropic soft biological tissues. This approach presents an advantage over the Lagrangian formulation, because it allows for arbitrary choices of reference and intermediate configurations.

Pandolfi *et al*. [31] provide finite-element simulations to analyse the mechanical response of the human cornea. They develop and rely on a microstructural model of crosslink interaction between collagen fibrils and manage to obtain results and predictions for the localized thinning and bulging typically observed in keratoconus corneas.

Anisotropy can come in many forms: it can be due to the presence of not only stiff fibres embedded into a softer matrix but also residual stresses in the material. This type of stress is ubiquitous in nature and can be the result of a deformation and also of an irreversible process which is very hard to model (temperature or plastic changes, growth, remodelling, ageing, etc.) Riccobelli *et al*. [32] look at a constitutive modelling of this type of anisotropy with a strain energy density depending on both the deformation gradient and the initial stress tensor. In this modelling, the residual stress is given without reference to its origin or generation, and physically based restrictions are imposed and analysed on the strain energy function.

5. Superposition of small deformations on finite deformations, stability

There is a strong connection between the theory of 'Small deformations superimposed on finite deformations' and the study of stability.

Stability theory is a difficult topic in any field of mechanics. Classical mechanics shows how subtle the concepts of stability are. In continuum mechanics, much has been done on stability in fluid mechanics, in contrast to the study of stability in solids, where very little general progress can be achieved due to the huge variety of constitutive laws available. One way to provide generality is to study universal solutions of nonlinear elasticity and to then linearize the equations in the neighbourhood of the large solution. This is the essence of Rivlin's small-on-large method: once a family of small-amplitude solutions is found, it can be argued that Euler's criterion of instability is met [33]. This approach was pioneered by Green & Zerna [34], Biot [35] and Hayes & Rivlin [36]. Alawiye *et al*. [37] follow this tradition by looking at the wrinkling that occurs on the surface of a compliant half-space covered with a stiffer layer. They are able to derive the conditions for surface instability when the deformation is driven by applied compressive forces and also by growth, for a large variety of boundary and material effects.

Of course, there are many other criteria for instability in nonlinear elasticity, and revealing connections to be made with the topics of energy, convexity, infinitesimal stability, strong ellipticity and uniqueness. For instance, the study of the 'Rivlin cube' problem reveals that at least seven equilibrated homogeneous deformations exist when equal dead loads are applied on the six faces of a neo-Hookean cube, some stable, some unstable [38]. In this collection, Mihai *et al*. [39] revisit the problem of Rivlin's cube by considering its stochastic version—in the sense that the elastic parameters are random variables because of uncertainties—and look at the effects of stochasticity on bifurcation diagrams.

6. Non-Newtonian fluids

Rivlin took a two-step approach to study non-Newtonian fluids. First, he investigated the equivalence between non-Newtonian fluids and turbulence theory; then he considered secondary flows. Since then, the explosion of computational methods and tools in fluid mechanics seems to have rendered analytical methods and the key role of modelling almost redundant, but an analytical path [40,41] based on Rivlin's approach surely needs reviving.

The connection between non-Newtonian fluids and turbulent flows has generated a long discussion with origins tracing back to the early papers of Rivlin in 1947. However, even the more recent nonlinearly dispersive Navier–Stokes-alpha (NS-α) model of incompressible fluid turbulence (also called the viscous Camassa–Holm equations) makes a link [42] with the second-grade fluid theory of Rivlin and co-workers.

The starting point of secondary flow studies is a paper by Fosdick *et al*. [43], showing that secondary flows are necessary in general situations for fluids. A parallel effort can be found in elasticity [44] and in turbulent flows [45]. In fact, secondary motions are more ubiquitous than was envisaged. The semi-inverse method championed by Rivlin to find analytical solutions to the field equations of continuum mechanics gives rise to many *compatibility* problems and opens the door to many perturbative schemes connected with the appearance of the secondary deformations or motions.

Here, Gomez-Constante & Rajagopal [46] study the flow of a certain type of non-Newtonian fluids in tubes of elliptic and other non-circular cross-sections. They set out to determine the velocity field and the stresses generated at the tube's boundary. They also provide a critical appraisal of Rivlin's work in this area.

7. Electro-magnetism

Rivlin's work on electro-magnetism forms the basis of our modern understanding of soft electro-active materials, which are of singular current interest in both academia and industry, with potential applications as soft robots, artificial muscles, energy harvesters and flexible electronics. The rigorous modelling of these materials is partly based on the foundations developed by Rivlin and his co-workers more than 50 years ago.

For a long time confined to modelling hard electro-active materials, the subject has undergone a revival in the past 15 years or so, due to the availability of inexpensive soft dielectric membranes capable of very large mechanical actuations when subject to large voltages. Their applicability is still limited because the voltages required are in the tens of thousands of volts. To reduce the voltage, one can reduce the thickness of the membrane, now typically in the order of the millimetre or less. However, as the membrane expands with the actuation, its thickness reduces by incompressibility and electrical breakdown becomes a real and common threat. Dorfmann & Ogden [47] use the theory of incremental elasticity (small-on-large, see §5) to study the stability of thin electro-active plates and of a thin-walled tube made of an electro-elastic material. They analyse, in turn, the possibility of electro-mechanical breakdown due to the Hessian matrix associated with the free energy of the system losing its positive definiteness, or due to the appearance of small-amplitude inhomogeneous wrinkles on the faces of the plates and tube.

8. Waves in viscoelastic materials

Elastic wave propagation in solids has a long history, especially in the acoustics of crystals. In the early part of the twentieth century, attention turned to waves travelling in a *deformed* solid, and thus subject to strain-induced anisotropy. Here, some great progress was made by Hadamard [48] and by Brillouin [49]. The former constructed a model for the Aether, which allows for any plane homogeneous wave of arbitrary amplitude to propagate in every direction once the medium is deformed homogeneously. The latter established the general equations of acousto-elasticity, which connects a pre-deformation (or equivalently, a pre-stress) to small-amplitude elastic wave propagation; specifically, he treated the problem of homogeneous plane waves in a general elastic solid of infinite extent, subject to a small pre-deformation ('small-on-small', so to speak).

Hayes & Rivlin [36] were able to go further, by looking at surface Rayleigh and Love waves on elastic half-spaces subject to large pre-strains. They also looked at the implications of having a real speed of propagation, and ensuring it is not zero, for strong ellipticity and stability (see §5). Their papers are still used, to date, in geophysics and in non-destructive evaluation of strained hard solids; see, for instance, the handbooks [50,51]. However, it is only recently that their formulae (or rather, their modern version, due to Chadwick & Ogden [52]) have been used experimentally for soft matter, in parallel with the development of imaging techniques of ultrasonic shear wave elastography, based on the elastic Cherenkov effect in isotropic [53] and anisotropic [54] soft gels and tissues.

Li *et al*. [55] present an in-depth investigation of surface wave propagation based on ultrasonic elastography. The difficulty is that their medium, a soft gel, presents strong heterogeneity with depth, as can be expected in several natural and engineering settings. They are nonetheless able to provide a complete theoretical, computational and experimental treatment of the problem.

Parnell & De Pascalis [56] also look at small-amplitude waves in inhomogeneous solids. They include the effects of viscosity, pre-stress and homogeneization, to study how solids can be designed to behave as metamaterials for acoustic waves.

Data accessibility. This article has no additional data.
Competing interests. We declare we have no competing interests.
Funding. No funding has been received for this article.

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