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Letter to the Editor

Explicit secular equation for Scholte waves over a monoclinic crystal

M. Destrade*
Laboratoire de Modélisation en Mécanique, CNRS, Université Pierre et Marie Curie (UMR 7607), 4 place Jussieu, Tour 66, case 162, 75252 Paris, Cedex 05, France

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1. Introduction

Scholte waves are acoustic waves propagating at a fluid/solid interface. They are localized in the neighborhood of the phase boundary in the sense that they decay exponentially in both directions along the normal to the interface. Johnson [1] established the explicit secular equation for Scholte waves over an orthorhombic crystal. In his case, the crystal is cut along a plane $x_2 = 0$ containing *two* crystallographic axes Ox_1 and Ox_3; the wave propagates with speed v in the x_1 direction; the solid is characterized by a mass density ρ_s and relevant elastic stiffnesses C_{11}, C_{12}, C_{22} and C_{66}; the fluid by a mass density ρ_f and speed of sound c. The secular equation is

$$Z\sqrt{C_{11}C_{22} - C_{12}^2 - 2C_{12}C_{66} - (C_{22} + C_{66})X + 2\sqrt{C_{22}C_{66}(C_{11} - X)(C_{66} - X)}} \\ - \sqrt{\frac{C_{66} - X}{C_{11} - X}}(C_{11}C_{22} - C_{12}^2 - C_{22}X) + X\sqrt{C_{22}C_{66}} = 0, \tag{1.1}$$

where

$$X = \rho_s v^2, \qquad Z = \frac{\rho_f v^2}{\sqrt{1 - v^2/c^2}}. \tag{1.2}$$

For instance, consider a frozen lake with a layer of ice assumed thick enough to be considered as a semi-infinite body. At 0.01°C under 1 bar, the density of water is [2]: $\rho_f = 999.84\ \text{kg/m}^3$ and sound propagates at $c = 1402.4$ m/s; the second line of Table 1 lists the elastic stiffnesses and density of ice [3]; according to Eq. (1.1), Scholte waves propagate for this model at speed $v_S = 1237.6$ m/s. Ice however has the special property of being transversally isotropic, which means that any plane containing the x_3-axis is a symmetry plane and so the speed v_S is the same for any orientation of the water/ice interface plane containing the x_3-axis.

*Tel.: +33-1-4427-7166; fax: +33-1-4427-5259.
E-mail address: destrade@lmm.jussieu.fr (M. Destrade).

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Table 1
Values of the elastic stiffnesses (10^{10} N/m^2), density (kg/m^3), and surface (Rayleigh) wave speed (m/s) for three crystals

Crystal	C_{11}	C_{22}	C_{12}	C_{16}	C_{26}	C_{66}	ρ_s	v_R
Ice (−5°C)	1.38	1.38	0.707	0	0	0.3365	940	1766
Gypsum	50.2	94.5	28.2	−7.5	−11.0	32.4	2310	3011
Terpine	1.25	0.99	0.38	0	0	0.346	1110	1644
Germanium	12.92	12.92	4.79	0	0	6.70	5320	2936

The aim of this Letter to the Editor is to derive explicitly the secular equation for Scholte waves at the interface between a fluid and an anisotropic crystal cut along a plane containing the normal to a single symmetry plane, that is containing only *one* crystallographic axis. In effect, the crystal may be a monoclinic crystal with symmetry plane at $x_3 = 0$, or a rhombic, tetragonal or cubic crystal cut along a plane containing x_3 and making an angle $\theta \neq 0$ with the other crystallographic planes; the higher symmetry cases ($\theta = 0$ or transversally isotropic and isotropic crystals) are covered by Eq. (1.1). For cases with less symmetries, one can turn to approximate solutions [4] as long as the anisotropy is weak.

2. Equations of motion and boundary conditions

Consider two half-spaces delimited by the plane $x_2 = 0$; the upper one $x_2 < 0$ is filled with an inviscid fluid, the lower one $x_2 > 0$ is made of a monoclinic crystal with symmetry plane at $x_3 = 0$ whose relevant non-zero reduced compliances are s'_{11}, s'_{22}, s'_{12}, s'_{16}, s'_{26} and s'_{66}. At the interface, an inhomogeneous plane wave travels with speed v and wave number k in the x_1 direction, and decays rapidly in the $x_2 \to \pm\infty$ directions.

In the *solid*, the corresponding equations of motion are written as a first-order differential system for the 4-component displacement-traction vector,

$$\xi' = \mathrm{i}\mathbf{N}\xi, \quad \xi(kx_2) = [U_1(kx_2), U_2(kx_2), t_{12}(kx_2), t_{22}(kx_2)]^{\mathrm{T}}, \tag{2.1}$$

where the functions U_i and t_{i2} are related to the in-plane mechanical displacements u_1, u_2 and in-plane tractions σ_{12}, σ_{22} through

$$u_i(x_1, x_2, x_3, t) = U_i(kx_2)\mathrm{e}^{\mathrm{i}k(x_1 - vt)}, \quad \sigma_{i2}(x_1, x_2, x_3, t) = \mathrm{i}kt_{i2}(kx_2)\mathrm{e}^{\mathrm{i}k(x_1 - vt)}. \tag{2.2}$$

In Eq. (2.1), the (4×4) matrix $\mathbf{N}$ is given by [5,6]

$$\mathbf{N} = \begin{bmatrix} -r_6 & -1 & n_{66} & n_{26} \\ -r_2 & 0 & n_{26} & n_{22} \\ X - \eta & 0 & -r_6 & -r_2 \\ 0 & X & -1 & 0 \end{bmatrix}, \tag{2.3}$$

where $X = \rho_s v^2$ and

$$\eta = \frac{1}{s'_{11}}, \quad r_i = -\frac{s'_{1i}}{s'_{11}}, \quad n_{ij} = \frac{1}{s'_{11}} \begin{vmatrix} s'_{11} & s'_{1j} \\ s'_{1i} & s'_{ij} \end{vmatrix}. \tag{2.4}$$

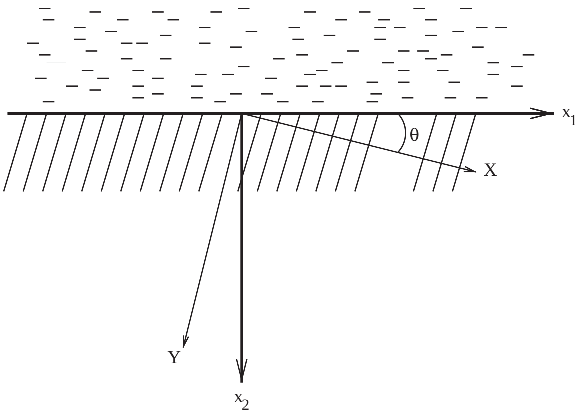

Fig. 1. Fluid/solid interface.

These equations also cover the case of a wave (2.2) travelling in a crystal of rhombic, tetragonal or cubic symmetry, with acoustic axes XYx_3 and reduced compliances S'_{ij}, cut along the plane $x_2 = 0$ containing the x_3-axis and making an angle θ with the crystallographic XY plane (see Fig. 1). In that case, the reduced compliances s'_{ij} along the x_i-axis are given in terms of those along the crystallographic axes XYx_3 by (see [7]),

$$\begin{aligned}
s'_{11} &= S'_{11}\cos^4\theta + (2S'_{12} + S'_{66})\cos^2\theta\sin^2\theta + S'_{22}\sin^4\theta,\\
s'_{22} &= S'_{22}\cos^4\theta + (2S'_{12} + S'_{66})\cos^2\theta\sin^2\theta + S'_{11}\sin^4\theta,\\
s'_{12} &= S'_{12} + (S'_{11} + S'_{22} - 2S'_{12} - S'_{66})\cos^2\theta\sin^2\theta,\\
s'_{66} &= S'_{66} + 4(S'_{11} + S'_{22} - 2S'_{12} - S'_{66})\cos^2\theta\sin^2\theta.\\
s'_{16} &= [2S'_{22}\sin^2\theta - 2S'_{11}\cos^2\theta + (2S'_{12} + S'_{66})(\cos^2\theta - \sin^2\theta)]\cos\theta\sin\theta,\\
s'_{26} &= [2S'_{22}\cos^2\theta - 2S'_{11}\sin^2\theta - (2S'_{12} + S'_{66})(\cos^2\theta - \sin^2\theta)]\cos\theta\sin\theta.
\end{aligned} \tag{2.5}$$

Note that for transversally isotropic crystals, the following relationships hold, $S'_{11} = S'_{22}$, $S'_{66} = 2(S'_{11} - S'_{12})$, and the rotation does not affect the values of the compliances ($s'_{ij} = S'_{ij}$). Destrade [8] recently showed that for waves vanishing with increasing distance from the plane $x_2 = 0$, the following fundamental relationships hold for any positive or negative integer power n of the matrix $\mathbf{N}$:

$$\bar{\boldsymbol{\xi}}(0)\cdot\hat{\mathbf{I}}\mathbf{N}^n\boldsymbol{\xi}(0) = 0, \quad \text{where} \quad \hat{\mathbf{I}} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.6}$$

Because of the Cayley–Hamilton theorem, only three consecutive powers of $\mathbf{N}$ are linearly independent so that Eq. (2.6) reduces to only three linearly independent equations.

In the *fluid*, the normal displacement and the normal stress component are connected, as recalled by Barnett et al. [9], by the (real) normal impedance Z defined in Eq. $(1.2)_2$,

$$\sigma_{22} = kZu_2. \tag{2.7}$$

At the *solid/fluid interface*, the normal displacement and the normal stress component are continuous, and the shear stress component is zero. It follows from these boundary conditions and from Eqs. $(2.1)_2$, (2.2) and (2.7), that the displacement-traction vector at the interface $x_2 = 0^+$ is of the form

$$\xi(0^+) = U_2(0)[\alpha, 1, 0, -\mathrm{i}Z]^{\mathrm{T}}, \tag{2.8}$$

where $\alpha = U_1(0^+)/U_2(0)$.

Now the fundamental equations (2.6) read

$$(N^n)_{32}(\alpha + \bar{\alpha}) + \mathrm{i}Z(N^n)_{21}(\alpha - \bar{\alpha}) + (N^n)_{31}\alpha\bar{\alpha} = -(N^n)_{42} - Z^2(N^n)_{24}. \tag{2.9}$$

Writing α as $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ and taking in turn $n = -1, 1, 2$, a non-homogeneous linear system of equations follows:

$$\mathbf{Ab} = \mathbf{d}, \quad \mathbf{A} = \begin{bmatrix} N^*_{32} & ZN^*_{22} & N^*_{31} \\ 0 & ZN_{22} & N_{31} \\ (N^2)_{32} & Z(N^2)_{22} & (N^2)_{31} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2\alpha_1 \\ -2\alpha_2 \\ \alpha_1^2 + \alpha_2^2 \end{bmatrix}, \quad \mathbf{d} = -\begin{bmatrix} N^*_{42} + Z^2N^*_{24} \\ N_{42} + Z^2N_{24} \\ Z^2(N^2)_{24} \end{bmatrix}, \tag{2.10}$$

where $\mathbf{N}^*$ denotes the adjoint of $\mathbf{N}$. The unique solutions to the system are $b_k = \Delta_k/\Delta$, where $\Delta = \det \mathbf{A}$ and Δ_k is the determinant of the matrix derived from $\mathbf{A}$ by replacing the kth column with $\mathbf{d}$. However, the b_k are linked by $b_1^2 + b_2^2 = 4b_3$, which is the explicit secular equation for Scholte wave over a monoclinic crystal with symmetry plane at $x_3 = 0$,

$$\Delta_1^2 + \Delta_2^2 = 4\Delta\Delta_3. \tag{2.11}$$

As a check, the limit case of a solid/vacuum interface is examined. When the density of the fluid ρ_f is taken as zero, then by $(1.2)_2$ $Z = 0$, and so $\Delta = \Delta_1 = \Delta_3 = 0$. The secular equation reduces to $\Delta_2 = 0$ (written at $Z = 0$), that is the following quartic in $X = \rho_s v^2$ [10,5,6],

$$\begin{vmatrix} X[r_2r_6 - n_{26}(X-\eta)] & (X-\eta)(1+n_{66}X) + r_6^2X & X[r_2^2 - n_{66}(X-\eta)] \\ 0 & X & X-\eta \\ (1+r_2)X-\eta & 0 & 2r_6(X-\eta) \end{vmatrix} = 0. \tag{2.12}$$

3. Examples

Calculations for usual combinations of a solid and a fluid show that in general the speed of a Scholte wave is very close to the speed of sound in the fluid. Hence, consider water ($\rho_f = 1025$ kg/m^3, $c = 1531$ m/s at 25°C [11]) over gypsum (monoclinic, ρ_s and C_{ij} in Table 1 [12]): the secular equation (2.11) yields a Scholte wave speed within the interval (1519 and 1526 m/s) (depending on the orientation of the cut plane), which is within less than 0.8% of the speed of sound in the water.

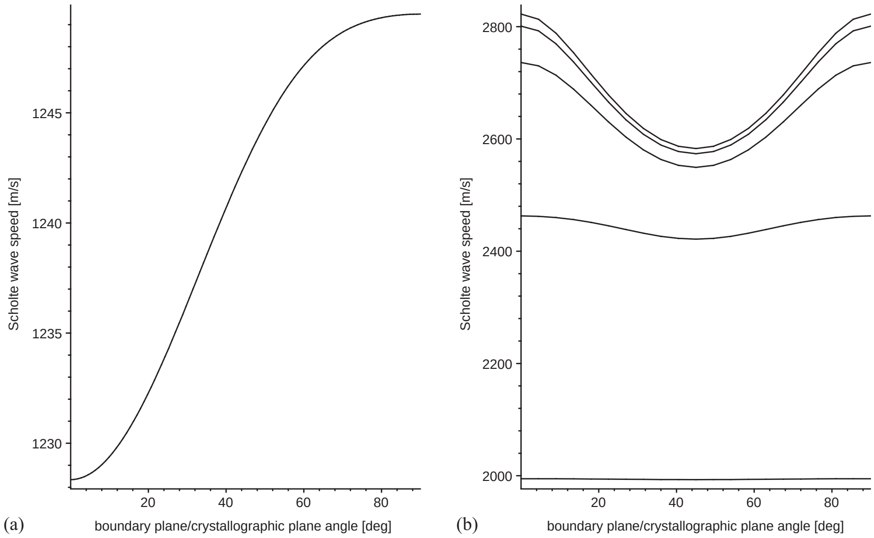

Fig. 2. Scholte wave speeds for (a) Water/Terpine interface and (b) Methanol/Germanium interface, where the speed of sound in the fluid is (m/s): 2000 (bottom curve), 2500, 3000, 3500 and 4000 (top curve).

Yet for certain choices, the Scholte wave speed moves away from the speed of sound in the fluid. One example is the combination ice/water presented in the Introduction. A second example is the combination of pure water ($\rho_f = 998$ kg/m^3, $c = 1498$ m/s at 25°C [11]) and Terpine Monohydride (orthorhombic, ρ_s and C_{ij} in Table 1 [3]): at $\theta = 0°$ and $90°$ (crystal cut along a plane containing two crystallographic axes) the wave propagates at 1228.3 and 1249.5 m/s, respectively; Fig. 2(a) shows how the Scholte wave speed varies between these two extremes as a function of θ. Another way of separating distinctly the Scholte wave speed from the sound speed is to increase the pressure, and hence the speed of sound, in the fluid. Crowhurst et al. [13] recently measured the Scholte wave speed for Methanol over Germanium in a diamond anvil cell: as the pressure increases from 0.56 to 2.2 GPa, so does the speed of sound in Methanol, from about 2500 to 3500 m/s. In Table 1, the stiffnesses and density of Germanium (cubic) at 20° are recalled [3]; the density of Methanol is 791.4 kg/m^3 at 20° [11]. Fig. 2(b) shows, in agreement with their results, the combined influence of orientation and speed of sound on Scholte wave propagation; each curve corresponds to a different speed of sound in Methanol, from $c = 2000$ (bottom curve) to 4000 m/s (top curve) by 500 m/s increments.

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