



Surface waves in a deformed isotropic hyperelastic material subject to an isotropic internal constraint

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Received 29 July 2003; received in revised form 11 September 2003; accepted 25 September 2003

Available online 23 July 2004

Abstract

An isotropic elastic half-space is prestrained so that two of the principal axes of strain lie in the bounding plane, which itself remains free of traction. The material is subject to an isotropic constraint of arbitrary nature. A surface wave is propagated sinusoidally along the bounding surface in the direction of a principal axis of strain and decays away from the surface. The exact secular equation is derived by a direct method for such a principal surface wave; it is cubic in a quantity whose square is linearly related to the squared wave speed. For the prestrained material, replacing the squared wave speed by zero gives an explicit bifurcation, or stability, criterion. Conditions on the existence and uniqueness of surface waves are given. The bifurcation criterion is derived for specific strain energies in the case of four isotropic constraints: those of incompressibility, Bell, constant area, and Ericksen. In each case investigated, the bifurcation criterion is found to be of a universal nature in that it depends only on the principal stretches, not on the material constants. Some results related to the surface stability of arterial wall mechanics are also presented.

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Keywords: Surface waves; Finite deformation; Internal constraints; Bifurcation criterion

1. Introduction

There has been a large body of literature on surface waves propagating sinusoidally along the bounding planar surface of an elastic half-space with attenuation of the wave amplitude in the direction normal to the bounding plane. The earliest paper is that of Rayleigh [1], who investigated surface waves propagating across the surface of an isotropic earth in the context of seismology. The modern theory of surface waves derives in large measure from the sextic formalism of Stroh [2]. This approach has been employed by many different authors to address many different problems of linear anisotropic elasticity and the results have been comprehensively reviewed by Ting [3]. In parallel, tremendous progress has been made in the theory of small-amplitude surface waves propagating on finitely deformed, nonlinearly elastic half-spaces, from the seminal works of Hayes and Rivlin [4] (compressible materials) to those of Dowaikh and Ogden [5] and many others.

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In the present paper we consider an isotropic elastic half-space prestrained so that two of the principal axes of strain lie in the bounding plane which itself remains free of traction. Additionally, the material is subject to an isotropic constraint of an *arbitrary* nature. One example of such a constraint is incompressibility, which has received much attention in the literature, and another is the Bell constraint, whose experimental and theoretical properties have been thoroughly reviewed by Beatty [6, Chapter 2]. A wave is propagated sinusoidally in the direction of one of the principal axes lying in the bounding plane, has no in-plane displacement component in the direction of the second principal axis lying in this plane, but has attenuating amplitude in the direction of the third principal axis, orthogonal to this plane (principal surface wave). The theoretical framework for the study of these waves is setup in Section 2, including a discussion of four examples of isotropic constraints, the incremental equations of motion and of constraint, and the strong ellipticity condition.

In Section 3 we derive an explicit secular equation for these principal surface waves that is cubic in a quantity whose square is linearly related to the squared wave speed. On restricting attention to an unstrained isotropic material we find that the secular equation reduces to that found by Rayleigh [1] in the incompressible case. We show that for the unstrained material all isotropic constraints are the same, that is all reduce to incompressibility [7]. Returning to the general secular equation of the prestrained material and replacing the squared wave speed by zero we obtain an explicit bifurcation, or stability, criterion for the material. Some results on the existence and uniqueness of surface waves are given.

Finally, in Section 4 we examine the bifurcation criterion for each of four examples of isotropic constraint and obtain explicit results by choosing special forms of the strain energy function. In all cases considered we find that the bifurcation criterion is of a universal nature in that it depends only on the principal stretches, not on the material constants.

2. Preliminaries

2.1. Deformed constrained half-space with a free plane surface

We consider a semi-infinite body made of homogeneous isotropic hyperelastic material at rest in a configuration \mathcal{B}_u with strain energy density W per unit volume of \mathcal{B}_u and mass density ρ . Let (O, X_1, X_2, X_3) be a fixed rectangular Cartesian coordinate system such that the body occupies the region $X_2 \geq 0$. The orthonormal set of vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is aligned with the coordinate axes.

Loads P_1, P_2, P_3 are applied at infinity to deform and maintain the half-space in a static state \mathcal{B}_e of finite pure homogeneous deformation, with corresponding stretch ratios $\lambda_1, \lambda_2, \lambda_3$ in the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ directions. Thus, the position of a particle at (X_1, X_2, X_3) in \mathcal{B}_u is at $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in \mathcal{B}_e , where $\bar{x}_1 = \lambda_1 X_1, \bar{x}_2 = \lambda_2 X_2, \bar{x}_3 = \lambda_3 X_3$. The constant deformation gradient associated with the deformation is

$$\bar{\mathbf{F}} = \lambda_1 \mathbf{i} \otimes \mathbf{i} + \lambda_2 \mathbf{j} \otimes \mathbf{j} + \lambda_3 \mathbf{k} \otimes \mathbf{k}. \quad (2.1)$$

In an isotropic hyperelastic material the strain energy is a symmetric function $W(\lambda_1, \lambda_2, \lambda_3)$ of the principal stretches, i.e. its value is left unchanged by any permutation of the stretches $\lambda_1, \lambda_2, \lambda_3$. The material is subject to an isotropic internal constraint, written as

$$\Gamma(\lambda_1, \lambda_2, \lambda_3) = 0, \quad (2.2)$$

in which Γ is a symmetric function of the principal stretches λ_i . We restrict attention to constraints such that:

$$\Gamma_i > 0, \quad \text{where } \Gamma_i := \frac{\partial \Gamma}{\partial \lambda_i}. \quad (2.3)$$

Four examples of such constraints are treated explicitly in this paper and they are henceforward denoted by roman numerals: the incompressibility (I), Bell (II), areal (III), and Ericksen (IV) constraints,

$$\begin{aligned} \Gamma^I &:= \lambda_1 \lambda_2 \lambda_3 - 1 = 0, & \Gamma^{II} &:= \lambda_1 + \lambda_2 + \lambda_3 - 3 = 0, & \Gamma^{III} &:= \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 - 3 = 0, \\ \Gamma^{IV} &:= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 = 0. \end{aligned} \tag{2.4}$$

The incompressibility constraint is often used for the modelling of finite deformations of rubber-like materials and shows good correlation with experiment (see for instance [8, Chapter 7]). The Bell constraint was found to hold experimentally over countless trials on polycrystalline annealed solids, including aluminum, brass, copper, and mild steel, see [6, Chapter 2]. The areal (or constant area) constraint has the interpretation that a material cube in the reference configuration with edges parallel to the principal axes of strain retains the same total surface area after deformation; it was studied from a purely mathematical point of view by Bosi and Salvatori [9]. Finally, the fourth constraint was proposed by Ericksen [10] to model the behaviour of certain twinned elastic crystals. Although Ericksen proposed a multi-constrained model, pursued by Scott [11] in the context of wave propagation, some authors [12,13] refer to the single constraint (2.4)₄ in nonlinear elasticity theory as ‘Ericksen’s constraint’.

The general constraint (2.2) generates the workless reaction tensor $\bar{\mathbf{N}}$, see [14], given by $\bar{\mathbf{N}} = J^{-1} \bar{\mathbf{F}}(\partial \Gamma / \partial \bar{\mathbf{F}})^T$, where $J = \lambda_1 \lambda_2 \lambda_3$. Explicitly, the non-zero components of $\bar{\mathbf{N}}$ are

$$\bar{N}_{ii} = J^{-1} \lambda_i \Gamma_i \quad (\text{no sum}). \tag{2.5}$$

Thus for the four examples of constraints (2.4) we find the following constraint tensors:

$$\bar{\mathbf{N}}^I = \mathbf{1}, \quad \bar{\mathbf{N}}^{II} = J^{-1} \mathbf{V}, \quad \bar{\mathbf{N}}^{III} = J^{-1}[(\text{tr } \mathbf{V}) \mathbf{V} - \mathbf{V}^2], \quad \bar{\mathbf{N}}^{IV} = 2J^{-1} \mathbf{V}^2, \tag{2.6}$$

in terms of the left stretch tensor $\mathbf{V} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Associated with the deformation $\mathcal{B}_u \rightarrow \mathcal{B}_e$ is the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$ which takes diagonal form with non-zero components

$$\bar{\sigma}_{ii} = J^{-1} \lambda_i W_i + \bar{P} \bar{N}_{ii} \quad (\text{no sum}), \tag{2.7}$$

where $W_i := \partial W / \partial \lambda_i$ and \bar{P} is a scalar to be determined from the equations of equilibrium and boundary conditions as follows. First, we note that for \bar{P} constant, the equations of equilibrium $\partial \bar{\sigma}_{ij} / \partial \bar{x}_j = 0$ are automatically satisfied. Next, we assume that the surface $\bar{x}_2 = 0$ is free of tractions; it follows that $\bar{\sigma}_{22} = 0$ (and $P_2 = 0$) and so

$$\bar{P} = -J^{-1} \lambda_2 \frac{W_2}{\bar{N}_{22}} = -\frac{W_2}{\Gamma_2}. \tag{2.8}$$

Consequently, the constant loads P_1, P_3 needed at infinity in order to maintain the half-space in the deformed configuration \mathcal{B}_e are ($k = 1, 3$, no sum),

$$P_k = -\bar{\sigma}_{kk} = \left(\frac{W_2}{\Gamma_2} \right) \bar{N}_{kk} - J^{-1} \lambda_k W_k = (W_2 \Gamma_k - W_k \Gamma_2) \frac{\lambda_k}{J \Gamma_2}. \tag{2.9}$$

Note that when the boundary surface is not free of tractions ($\bar{\sigma}_{22} \neq 0$) then \bar{P} in (2.8) must be replaced by $\bar{P} = (J \bar{\sigma}_{22} - \lambda_2 W_2) / \lambda_2 \Gamma_2$. However, the primary interest of this paper is in studying the influence of internal constraints upon the propagation of surface waves rather than the influence of pre-loading, and we assume henceforward that (2.8) holds.

2.2. Superposed infinitesimal motion and strong ellipticity condition

We now consider the propagation of an incremental motion in the deformed half-space $\mathcal{B}_e \rightarrow \mathcal{B}_t$, described by

$$\mathbf{x} = \bar{\mathbf{x}} + \epsilon \mathbf{u}(\bar{\mathbf{x}}, t), \tag{2.10}$$

where \mathbf{x} is the position in \mathcal{B}_t of a particle which was at $\bar{\mathbf{x}}$ in \mathcal{B}_e and \mathbf{u} is referred to as the displacement vector. The parameter ϵ is small, so that terms of order higher than one in ϵ may be neglected. The linear fourth-order instantaneous elasticity tensor \mathbf{B}^* associated with the motion is [14, (2.19)]

$$B_{ijkl}^* = B_{ijkl} + \bar{P}\check{B}_{ijkl} = B_{ijkl} - \left(\frac{W_2}{\Gamma_2}\right)\check{B}_{ijkl}, \tag{2.11}$$

where the non-zero components of \mathbf{B} and $\check{\mathbf{B}}$ are given by [8, (6.3.15)]

$$\begin{aligned} JB_{ijj} &= \lambda_i \lambda_j W_{ij}, & JB_{ijj} &= \frac{\lambda_i W_i - \lambda_j W_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2, & JB_{iji} &= JB_{ijj} - \lambda_i W_i, & J\check{B}_{ijj} &= \lambda_i \lambda_j \Gamma_{ij}, \\ J\check{B}_{ijj} &= \frac{\lambda_i \Gamma_i - \lambda_j \Gamma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2, & J\check{B}_{iji} &= J\check{B}_{ijj} - \lambda_i \Gamma_i, \end{aligned} \tag{2.12}$$

and there is no summation over i or j . Here JB_{ijj} and $J\check{B}_{ijj}$ are defined when $i \neq j, \lambda_i \neq \lambda_j$; in the case where $i \neq j, \lambda_i = \lambda_j$, they must be replaced by $JB_{ijj} = (1/2)(JB_{iii} - JB_{ijj} + \lambda_i W_i)$, $J\check{B}_{ijj} = (1/2)(J\check{B}_{iii} - J\check{B}_{ijj} + \lambda_i \Gamma_i)$, see [8, (6.3.16)]. Because of hyperelasticity the symmetries $B_{ijkl}^* = B_{klij}^*$ hold so that in particular we have

$$B_{ijj}^* = B_{jjj}^*, \quad B_{iji}^* = B_{jii}^* \quad (\text{no sum}). \tag{2.13}$$

The incremental nominal stress associated with this motion is \mathbf{s} given by [14, (3.10)]

$$s_{ij} = \bar{\sigma}_{ij} + B_{ijkl}^* u_{l,k} + p\bar{N}_{ij}, \tag{2.14}$$

where p represents the increment in \bar{P} .

The equations of motion, together with the incremental constraint, read, from [14, (3.13)],

$$\bar{\rho} u_{j,tt} = s_{ij,i}, \quad \bar{N}_{11} u_{1,1} + \bar{N}_{22} u_{2,2} + \bar{N}_{33} u_{3,3} = 0. \tag{2.15}$$

For future convenience, we assume that \mathbf{B}^* is strongly elliptic, that is,

$$B_{ijkl}^* m_i n_j m_k n_l > 0 \quad \text{for all } \mathbf{m}, \mathbf{n} \text{ such that } m_i \bar{N}_{ij} n_j = 0. \tag{2.16}$$

The vectors

$$\begin{aligned} \mathbf{m} &= \bar{N}_{11}^{-1/2} \cos \theta \mathbf{i} + \bar{N}_{22}^{-1/2} \sin \theta \mathbf{j} = J^{1/2} [(\lambda_1 \Gamma_1)^{-1/2} \cos \theta \mathbf{i} + (\lambda_2 \Gamma_2)^{-1/2} \sin \theta \mathbf{j}], \\ \mathbf{n} &= -\bar{N}_{11}^{-1/2} \sin \theta \mathbf{i} + \bar{N}_{22}^{-1/2} \cos \theta \mathbf{j} = J^{1/2} [-(\lambda_1 \Gamma_1)^{-1/2} \sin \theta \mathbf{i} + (\lambda_2 \Gamma_2)^{-1/2} \cos \theta \mathbf{j}], \end{aligned} \tag{2.17}$$

$0 \leq \theta \leq 2\pi$, are two vectors satisfying (2.16)₂. Introducing the quantities A, B, C , defined by

$$\begin{aligned} A &= (\lambda_1 \lambda_2 \Gamma_1 \Gamma_2)^{-1} B_{1212}^*, & C &= (\lambda_1 \lambda_2 \Gamma_1 \Gamma_2)^{-1} B_{2121}^*, \\ B &= \frac{1}{2} [(\lambda_1 \Gamma_1)^{-2} B_{1111}^* + (\lambda_2 \Gamma_2)^{-2} B_{2222}^*] - (\lambda_1 \lambda_2 \Gamma_1 \Gamma_2)^{-1} (B_{1122}^* + B_{1221}^*), \end{aligned} \tag{2.18}$$

we obtain from (2.16) the inequality

$$A \cos^4 \theta + 2B \sin^2 \theta \cos^2 \theta + C \sin^4 \theta > 0, \tag{2.19}$$

holding for all θ . In particular, the choices $\theta = 0, \pi/2$, $\arctan(A/C)^{1/4}$, give in turn

$$A > 0, \quad C > 0, \quad B + \sqrt{AC} > 0. \tag{2.20}$$

Using different notations, similar inequalities are given by Knowles and Sternberg [15] for unconstrained materials, by Ogden [8] for incompressible materials, and by Destrade [16] for Bell materials.

3. Surface waves

Here we specialize the equations of motion to the consideration of an inhomogeneous principal plane wave traveling over the free surface $\bar{x}_2 = 0$ and derive the corresponding secular equation.

3.1. Equations of motion and boundary conditions

A surface (Rayleigh) wave travels sinusoidally with time t in a direction parallel to the plane $\bar{x}_2 = 0$ ($X_2 = 0$) leaving it free of tractions and decaying away from this surface as $\bar{x}_2 \rightarrow \infty$. For simplicity, we consider a wave propagating with speed v and wave number k in the principal direction of prestrain $OX_1 = O\bar{x}_1$. We refer to this as a principal wave. For this wave, antiplane strain decouples from inplane strain, and the displacement components are of the form:

$$u_i = U_i(k\bar{x}_2) e^{ik(\bar{x}_1 - vt)} \quad (i = 1, 2), \quad u_3 = 0, \quad p = kQ(k\bar{x}_2) e^{ik(\bar{x}_1 - vt)}, \quad (3.1)$$

in which U_1 , U_2 and Q are functions of $k\bar{x}_2$ to be determined. Similarly, the antiplane stress decouples from the plane stress, and the equations of motion (2.15) reduce to

$$s_{11,1} + s_{21,2} = \bar{\rho}u_{1,t}, \quad s_{12,1} + s_{22,2} = \bar{\rho}u_{2,t}, \quad \bar{N}_{11}u_{1,1} + \bar{N}_{22}u_{2,2} = 0. \quad (3.2)$$

Because the $\bar{\sigma}_{ij}$ terms in the components (2.14) of s_{ij} are constant, we write the relevant stress components in the form

$$s_{ij} = \bar{\sigma}_{ij} + kS_{ij}(k\bar{x}_2) e^{ik(\bar{x}_1 - vt)} \quad (i, j = 1, 2), \quad (3.3)$$

and the equations of motion reduce further to

$$iS_{11} + S'_{21} = -\bar{\rho}v^2U_1, \quad iS_{12} + S'_{22} = -\bar{\rho}v^2U_2, \quad i\lambda_1\Gamma_1U_1 + \lambda_2\Gamma_2U'_2 = 0. \quad (3.4)$$

Explicitly, the S_{ij} are given from (2.14) and (3.3) by

$$\begin{aligned} S_{11} &= iB_{1111}^*U_1 + B_{1122}^*U'_2 + Q\bar{N}_{11}, & S_{12} &= B_{1221}^*U'_1 + iB_{1212}^*U_2, \\ S_{22} &= iB_{1122}^*U_1 + B_{2222}^*U'_2 + Q\bar{N}_{22}, & S_{21} &= B_{2121}^*U'_1 + iB_{1221}^*U_2, \end{aligned} \quad (3.5)$$

in which the symmetries (2.13) have been used.

Now we use (3.4) and (3.5) to formulate the problem as a system of four first-order differential equations for the unknown functions U_1 , U_2 , S_{21} , S_{22} :

$$\begin{aligned} U'_1 &= -i\frac{B_{1221}^*}{B_{2121}^*}U_2 + \frac{1}{B_{2121}^*}S_{21}, & U'_2 &= -i\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}U_1, \\ S'_{21} &= \left[B_{1111}^* - 2\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{1122}^* + \left(\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}\right)^2 B_{2222}^* - \bar{\rho}v^2 \right] U_1 - i\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}S_{22}, \\ S'_{22} &= \left[\frac{B_{2121}^*B_{1212}^* - B_{1221}^{*2}}{B_{2121}^*} - \bar{\rho}v^2 \right] U_2 - i\frac{B_{1221}^*}{B_{2121}^*}S_{21}. \end{aligned} \quad (3.6)$$

Eqs. (3.6)₁ and (3.6)₂ are obtained from (3.5)₄ and (3.4)₃, respectively, and (2.5) is employed in the latter. Eq. (3.6)₃ is obtained by eliminating Q between (3.5)₁ and (3.5)₃ using (2.5), and then eliminating S_{11} between this equation and (3.4)₁. Finally, (3.6)₄ is obtained by eliminating S_{12} between (3.4)₂ and (3.5)₂ and then using (3.5)₄ to eliminate U'_1 in favor of S_{21} .

Eq. (3.6) are subject to the boundary conditions of decay as $\bar{x}_2 \rightarrow \infty$ and of vanishing traction on $\bar{x}_2 = 0$:

$$S_{21}(0) = S_{22}(0) = 0. \quad (3.7)$$

Furthermore, each unknown may be written in terms of a single unknown function φ of the variable

$$z = \frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}k\bar{x}_2$$

with prime now denoting differentiation with respect to z :

$$\begin{aligned}
 U_1 &= i\varphi'(z), & U_2 &= \varphi(z), & S_{21} &= i\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{2121}^*\varphi''(z) + iB_{1221}^*\varphi(z), \\
 S_{22} &= -\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{2121}^*\varphi'''(z) + \frac{\lambda_2\Gamma_2}{\lambda_1\Gamma_1}\left[B_{1111}^* - \frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}(B_{1221}^* + 2B_{1122}^*) + \left(\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}\right)^2 B_{2222}^* - \bar{\rho}v^2\right]\varphi'(z). \tag{3.8}
 \end{aligned}$$

Here, the expressions for U_1 , S_{21} , and S_{22} were obtained from (3.6)_{2,1,3}, respectively. The last equation of this set, namely (3.6)₄, then yields a differential equation for $\varphi(z)$:

$$\gamma^*\varphi'''' - (2\beta^* - \bar{\rho}v^2)\varphi'' + (\alpha^* - \bar{\rho}v^2)\varphi = 0, \tag{3.9}$$

where

$$\begin{aligned}
 \alpha^* &= B_{1212}^*, & \gamma^* &= \left(\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}\right)^2 B_{2121}^* = \frac{\Gamma_1^2}{\Gamma_2^2}\alpha^*, \\
 2\beta^* &= B_{1111}^* - 2\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}(B_{1221}^* + B_{1122}^*) + \left(\frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}\right)^2 B_{2222}^*. \tag{3.10}
 \end{aligned}$$

We have used the property $\lambda_i^2 B_{ijji}^* = \lambda_j^2 B_{ijij}^*$ ($i \neq j$, no sum) derived from (2.12). By comparison with the quantities A , B , C , defined in (2.18), and use of the consequences (2.20) of the strong ellipticity (S–E) condition (2.16), we find that these coefficients satisfy the inequalities

$$\alpha^* > 0, \quad \gamma^* > 0, \quad \beta^* + \sqrt{\alpha^*\gamma^*} > 0. \tag{3.11}$$

3.2. Secular equation and bifurcation criterion

We now derive the exact form of the secular equation for a surface wave traveling in a principal direction of a deformed isotropic material subject to a single isotropic constraint. Because the wave amplitude decays as $z \rightarrow \infty$ away from the free plane $z = 0$, we seek a solution for φ in the form

$$\varphi(z) = A_1 e^{-s_1 z} + A_2 e^{-s_2 z}, \quad \Re(s_i) > 0, \quad s_1 \neq s_2. \tag{3.12}$$

From (3.9), the s_i are roots of the biquadratic

$$\gamma^*s^4 - (2\beta^* - \bar{\rho}v^2)s^2 + (\alpha^* - \bar{\rho}v^2) = 0. \tag{3.13}$$

The roots s_i^2 of this real quadratic are either both real (and, if so, both positive because we must have $\Re(s_i) > 0$) or they are a complex conjugate pair. In either case, $s_1^2 s_2^2 > 0$ and so, by (3.13),

$$0 \leq \bar{\rho}v^2 \leq \alpha^*. \tag{3.14}$$

Although necessary, this inequality on the squared wave speed is not sufficient to ensure the decay of the wave amplitude, as is seen in the next subsection.

The boundary conditions (3.7) yield, using (3.8)_{3,4}, (3.10) and (3.13),

$$\begin{aligned}
 \left(\gamma^*s_1^2 + \frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{1221}^*\right)A_1 + \left(\gamma^*s_2^2 + \frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{1221}^*\right)A_2 &= 0, \\
 s_1\left(\gamma^*s_2^2 + \frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{1221}^*\right)A_1 + s_2\left(\gamma^*s_1^2 + \frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{1221}^*\right)A_2 &= 0. \tag{3.15}
 \end{aligned}$$

The vanishing of the determinant of this homogeneous system gives the *secular equation*, that is, the equation for the wave speed. Introducing the quantities

$$\eta^* = \sqrt{\frac{\alpha^* - \bar{\rho}v^2}{\gamma^*}}, \quad \delta^* = \frac{\lambda_1\Gamma_1}{\lambda_2\Gamma_2}B_{1221}^* = \frac{\lambda_2\Gamma_2}{\lambda_1\Gamma_1}\gamma^*, \tag{3.16}$$

and using (3.13) we obtain from the vanishing of the determinant of (3.15),

$$f(\eta^*) := \eta^{*3} + \eta^{*2} + \frac{2\beta^* + 2\delta^* - \alpha^*}{\gamma^*} \eta^* - \frac{\delta^{*2}}{\gamma^{*2}} = 0, \tag{3.17}$$

after removing the $s_1 - s_2$ factor. Eq. (3.17) is the required secular equation.

In the special case where the biquadratic (3.13) has double roots, so that $s_2^2 = s_1^2$, the above derivation is no longer valid. The form (3.12) of solution must be replaced by

$$\varphi(z) = (A_1 + A_2 z) e^{-s_1 z}$$

but it can still be shown that the secular equation is given by (3.17). Thus (3.17) furnishes the secular equation in all cases.

The cubic equation (3.17) in η^* has remarkable features: the coefficients of the two highest powers are each always equal to unity, irrespective of the pre-deformation, constraint, and strain energy function. By (3.16)₃, the term independent of η^* depends only on λ_1, λ_2 , and the constraint, but *not* on the strain energy function. For instance, for the I–IV constraints (2.4), we find that

$$\frac{\delta^*}{\gamma^*} = 1, \quad \frac{\lambda_2}{\lambda_1}, \quad \frac{\lambda_1 + \lambda_3}{\lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_1}, \quad \frac{\lambda_2^2}{\lambda_1^2}, \tag{3.18}$$

respectively, whatever W may be. From (3.10)₂ it is clear also that α^*/γ^* is independent of W and it follows that the secular equation (3.17) depends on W only through the term β^*/γ^* appearing in the coefficient of the term linear in η^* .

In an undeformed material ($\lambda_1 = \lambda_2 = \lambda_3 = 1$), we have $\alpha^* = \beta^* = \gamma^* = \delta^* = \mu$, where μ is the infinitesimal shear modulus, and the secular equation (3.17) reduces to

$$f(\eta) := \eta^3 + \eta^2 + 3\eta - 1 = 0, \quad \eta = \sqrt{1 - \bar{\rho} \frac{v^2}{\mu}}. \tag{3.19}$$

The unique positive real root of this equation corresponds to $\bar{\rho} v^2 / \mu \approx 0.9126$, in accordance with Rayleigh’s result [1] for incompressible linear isotropic elastic materials.

Eq. (3.19) is valid for all isotropic constraints provided that the isotropic elastic material is undeformed. In fact, in the undeformed state we can show that all isotropic constraints are equivalent by arguing as follows (see Podio-Guidugli and Vianello [7] for an alternative treatment). The constraint is $\Gamma(\lambda_1, \lambda_2, \lambda_3) = 0$ where the function Γ is symmetric in its arguments, i.e. its value is left unchanged by any permutation of the stretches $\lambda_1, \lambda_2, \lambda_3$. For small strains write $\lambda_i = 1 + e_i$ where e_i is the extension ratio. The constraint holds for $\lambda_i = 1$ and for $\lambda_i = 1 + e_i$ (for each i) so an application of Taylor’s theorem gives

$$\frac{\partial \Gamma}{\partial \lambda_1} e_1 + \frac{\partial \Gamma}{\partial \lambda_2} e_2 + \frac{\partial \Gamma}{\partial \lambda_3} e_3 = 0,$$

where terms quadratic in e_i are neglected. The partial derivatives are evaluated at $\lambda_i = 1$ and, because of the symmetry condition on Γ , are all equal. Thus, in the undeformed state, each isotropic constraint takes the form

$$e_1 + e_2 + e_3 = 0 \tag{3.20}$$

of the constraint of incompressibility for infinitesimal deformations.

The *bifurcation criterion* is obtained by writing $v = 0$ in the secular equation (3.17) and indicates when the half-space might become unstable [17]:

$$f\left(\sqrt{\frac{\alpha^*}{\gamma^*}}\right) = \left(\frac{\alpha^*}{\gamma^*}\right)^{3/2} + \frac{\alpha^*}{\gamma^*} + \frac{2\beta^* + 2\delta^* - \alpha^*}{\gamma^*} \sqrt{\frac{\alpha^*}{\gamma^*}} - \frac{\delta^{*2}}{\gamma^{*2}} = 0, \tag{3.21}$$

which becomes

$$\frac{\Gamma_2^2}{\Gamma_1^2} + 2 \frac{\beta^* + \delta^*}{\gamma^*} \frac{\Gamma_2}{\Gamma_1} - \frac{\delta^{*2}}{\gamma^{*2}} = 0 \tag{3.22}$$

on using (3.10)₂ to eliminate α^*/γ^* . Using (2.11)₂, (2.12), (3.10) and (3.16)₂, we may rewrite (3.22) in terms of the derivatives with respect to λ_i of the strain energy function W and the constraint Γ :

$$\Gamma_2^2 W_{11} - 2\Gamma_1 \Gamma_2 W_{12} + \Gamma_1^2 W_{22} + \frac{\Gamma_1(\Gamma_2 W_1 - \Gamma_1 W_2)}{\lambda_1} - \frac{(\Gamma_{11} \Gamma_2^2 - 2\Gamma_1 \Gamma_2 \Gamma_{12} + \Gamma_{22} \Gamma_1^2) W_2}{\Gamma_2} = 0, \tag{3.23}$$

an explicit form of the bifurcation criterion.

3.3. Existence and uniqueness of a surface wave

Following Chadwick [18] we recast our secular equation (3.17) as

$$f(v) = \det \begin{bmatrix} \eta^* r^* & \frac{\delta^*}{\gamma^*} - \eta^* \\ \frac{\delta^*}{\gamma^*} - \eta^* & r^* \end{bmatrix} = 0, \quad \text{where } r^* = \sqrt{(\eta^* + 1)^2 - \frac{\alpha^* + \gamma^* - 2\beta^*}{\gamma^*}}. \tag{3.24}$$

Chadwick used this “matrix reformulation” of the secular equation in the case of incompressible (I) materials (where $\delta^*/\gamma^* = 1$) to derive, in a rigorous manner, essential results about the existence and uniqueness of a surface wave. For a general isotropic constraint (2.2), δ^*/γ^* is not necessarily equal to 1, but his method, together with the S–E inequalities (3.11), can nevertheless be directly transposed to our problem. In summary, we obtain the following results.

The limiting speed \hat{v} , which is the upper bound of the subsonic interval $I = [0, \hat{v}]$ for v where the decay requirement $\Re(s_i) > 0$ is satisfied, is defined by

$$\bar{\rho} \hat{v}^2 = \begin{cases} \alpha^* & \text{when } 2\beta^* > \alpha^*, \\ 2[\beta^* - \gamma^* + \sqrt{\gamma^*(\alpha^* - 2\beta^* + \gamma^*)}] < \alpha^* & \text{when } 2\beta^* < \alpha^*. \end{cases} \tag{3.25}$$

A necessary and sufficient condition of existence for a root v_R in I is

$$f(0) > 0, \quad f(\hat{v}) < 0. \tag{3.26}$$

Finally, when a root exists, it is unique.

4. The bifurcation criterion in special cases

We specialize the bifurcation criterion (3.23) to each of the four examples of isotropic constraints considered in this paper and pick special forms of the strain energy in order to obtain explicit results. In each case we find that the bifurcation criterion is of a (relative) universal nature in that it depends only on the principal stretches, not on the material constants.

4.1. Incompressibility (I)

For incompressibility, the bifurcation criterion (3.23) reduces to [5]:

$$\lambda_1^2 W_{11} - 2\lambda_1 \lambda_2 W_{12} + \lambda_2^2 W_{22} + \lambda_2 [W_1 + (2 - \lambda_1^{-1} \lambda_2) W_2] = 0. \tag{4.1}$$

We consider the strain energy function for “generalized Varga materials”,

$$W = d_1(\lambda_1 + \lambda_2 + \lambda_3 - 3) + d_2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 - 3), \tag{4.2}$$

in which d_1 and d_2 are material constants. The S–E conditions (3.11) lead to $d_1 + 2d_2\lambda_3 > 0$, and so $d_1 > 0, d_2 \geq 0$ or $d_1 \geq 0, d_2 > 0$. The bifurcation criterion (4.1) leads to

$$(d_1 + \lambda_3 d_2)(3\lambda_1 - \lambda_2) = 0, \quad \text{or} \quad \lambda_2 = 3\lambda_1, \tag{4.3}$$

independently of the choice of material constants.

4.2. Bell’s constraint (II)

For a Bell constrained material, (4.1) reduces to the bifurcation criterion [19]:

$$\lambda_1(W_{11} - 2W_{12} + W_{22}) + W_1 - W_2 = 0. \tag{4.4}$$

For the specific example of the “simple hyperelastic Bell material” [6] we take

$$W = d_2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 - 3) + d_3(\lambda_1\lambda_2\lambda_3 - 1). \tag{4.5}$$

Here the S–E conditions (3.11) lead to $-(d_2 + 2d_3\lambda_3) > 0$, and so $d_2 < 0, d_3 \leq 0$ or $d_2 \leq 0, d_3 < 0$, while the bifurcation criterion (4.1) reduces to

$$(d_2 + \lambda_3 d_3)(\lambda_2 - 3\lambda_1) = 0, \tag{4.6}$$

leading again to (4.3).

4.3. Areal constraint (III)

The bifurcation criterion (4.1) reduces to

$$\begin{aligned} &(\lambda_1 + \lambda_3)^2 W_{11} - 2(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)W_{12} + (\lambda_2 + \lambda_3)^2 W_{22} + \lambda_1^{-1}(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)W_1 \\ &+ (\lambda_2 + \lambda_3)[2 - \lambda_1^{-1}(\lambda_2 + \lambda_3)]W_2 = 0. \end{aligned} \tag{4.7}$$

For the following material, which we term “simple hyperelastic areal material”,

$$W = d_1(\lambda_1 + \lambda_2 + \lambda_3 - 3) + d_3(\lambda_1\lambda_2\lambda_3 - 1), \tag{4.8}$$

the S–E conditions (3.11) lead to $d_1 - d_3\lambda_3 > 0$, and so $d_1 > 0, d_3 \leq 0$ or $d_1 \geq 0, d_3 < 0$. The bifurcation criterion (4.7) reduces to

$$(d_1 - \lambda_3^2 d_3)(3\lambda_1 - \lambda_2) = 0, \tag{4.9}$$

leading once more to (4.3).

4.4. Ericksen’s constraint (IV)

For Ericksen materials the bifurcation criterion (4.1) reduces to

$$\lambda_2^2 W_{11} - 2\lambda_1\lambda_2 W_{12} + \lambda_1^2 W_{22} + \lambda_2[W_1 - \lambda_2^{-2}(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)W_2] = 0. \tag{4.10}$$

For the following “simple hyperelastic Ericksen material”,

$$W = D_2(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3) + D_3(\lambda_1^2\lambda_2^2\lambda_3^2 - 1), \tag{4.11}$$

Table 1
Critical stretch ratios $(\lambda_1)_{cr}$ for surface stability

	$\lambda_2 = \lambda_3$	$\lambda_3 = 1$	$\lambda_1 = \lambda_3$
Simple Bell	0.429	0.500	0.600
Simple areal	0.447	0.535	0.655
Generalized Varga	0.481	0.577	0.693
Simple Ericksen	0.603	0.658	0.730

the S–E conditions (3.11) lead to $-(D_2 + D_3\lambda_3^2) > 0$, and so $D_2 < 0$, $D_3 \leq 0$ or $D_2 \leq 0$, $D_3 < 0$. The bifurcation criterion (4.10) simplifies to

$$\lambda_1^3 + 5\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2 - \lambda_2^3 = 0. \quad (4.12)$$

4.5. Summary

For the four specific constraints and strain-energy functions above, we found relative universal bifurcation criteria, even though two independent material constants were involved. Moreover, the bifurcation criterion for each of the first three forms of W turns out to be the same, although the constraints are different; of course, the corresponding critical stretch ratios differ in each case because they are obtained by solving the bifurcation criterion in conjunction with the constraint condition. These differences are highlighted in Table 1, where the critical stretch ratios $(\lambda_1)_{cr}$ for surface stability in compression of the four constrained materials presented in this section have been computed in the case of plane strain $\lambda_3 = 1$, and of equibiaxial strains $\lambda_2 = \lambda_3$ and $\lambda_1 = \lambda_3$. All in all, it seems that simple hyperelastic Bell materials can be compressed the most before the bifurcation criterion is reached, whilst Ericksen materials of type (4.11) can be compressed the least.

As kindly pointed out by a referee, some questions worth investigating are raised by the previous results: what is the largest class of energy function for which the result is universal for any given constraint? for what energy functions is the bifurcation criterion the same for two (or more) different constraints (separately or together)? for a material whose energy function is linear in the invariants of the stretch tensor such as (4.2), (4.5), (4.8), which constraints lead to the same bifurcation criterion (4.3)? and so on. Unfortunately we must, because of space limitations, leave these problems open.

4.6. Concluding remark: an incompressible model for human thoracic aorta

We conclude with an example of a non-universal bifurcation criterion using a model taken from the biomechanics literature. Horgan and Saccomandi [20] recently discussed the merits of the “limiting chain extensibility model” proposed by Gent [21] and its applicability to the modelling of strain-stiffening biological tissues. For this incompressible (I) material, the strain energy is

$$W = \frac{1}{2}\mu J_m \ln \left(1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_m} \right), \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 < 3 + J_m, \quad (4.13)$$

where μ is the shear modulus and J_m is a constant. The smaller J_m is, the stiffer the material becomes; conversely the body becomes more deformable as J_m increases, with the neo-Hookean model as a limit for $J_m \rightarrow \infty$, where there are no restrictions on the values that the stretch ratios may take.

For the Gent model (4.13) the S–E conditions (3.11) lead to $\mu J_m > 0$ so that μ and J_m are of the same sign. The bifurcation criterion (4.1) yields

$$f(\lambda_1, \lambda_2)(J_m + 3 - \lambda_3^2) - f(\lambda_2, \lambda_1)(\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2) = 0, \quad (4.14)$$

where $f(x, y) := x^3 + x^2y + 3xy^2 - y^3$. Note that as $J_m \rightarrow \infty$ this equality tends to $f(\lambda_1, \lambda_2) = 0$, which is the universal bifurcation criterion for neo-Hookean (and Mooney–Rivlin) materials [17]. Using experimental data for the thoracic aorta of a 21-year-old male and of a 70-year-old male, Horgan and Saccomandi [20] found that for these samples, $J_m = 2.289, 0.422$, respectively. For these values of J_m and for the three special prestrains where $\lambda_3 = 1$, or $\lambda_2 = \lambda_3$, or $\lambda_1 = \lambda_3$, we find that there exists no value of λ_1 such that the bifurcation criterion is satisfied, indicating that the aorta of the two males is *always stable* near the surface with respect to finite compressions of these types. Keep in mind however that the range of possible ratios is limited by the inequality (4.13)₂. For example, in plane strain $\lambda_3 = 1$ (no extension in the X_3 direction), we find that

$$\sqrt{1 + \frac{J_m(1 - \sqrt{1 + 4J_m^{-1}})}{2}} < \lambda_{1,2} < \sqrt{1 + \frac{J_m(1 + \sqrt{1 + 4J_m^{-1}})}{2}}. \quad (4.15)$$

For the younger aorta, this range is [0.497, 2.010]; for the older, stiffer, aorta, the range is [0.727, 1.376].

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