Dedicated to Ray W. Ogden on the occasion of his 60th birthday

On interface waves in misoriented pre-stressed incompressible elastic solids

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Some relationships, fundamental to the resolution of interface wave problems, are presented. These equations allow for the derivation of explicit secular equations for problems involving waves localized near the plane boundary of anisotropic elastic half-spaces, such as Rayleigh, Scholte, or Stoneley waves. They are obtained rapidly, without recourse to the Stroh formalism. As an application, the problems of Stoneley wave propagation and of interface stability for misaligned predeformed incompressible half-spaces are treated. The upper and lower half-spaces are made of the same material, subject to the same prestress, and are rigidly bonded along a common principal plane. The principal axes in this plane do not, however, coincide, and the wave propagation is studied in the direction of the bisectrix of the angle between a principal axis of the upper half-space and a principal axis of the lower half-space.

Keywords: bimaterials; interface stability; pre-stress; Stoneley waves.

1. Introduction

We all know from experience or from intuition that when we press together two large, well-polished, glass plates, they will stick together extremely well; in fact we will have a tough job trying to separate them again because in effect, the two glass panes have become one. A somewhat similar process of 'gluing without glue' is used in the microelectronics and optoelectronics industries to bond together *semiconductor wafers* (Gösele & Tong, 1998). When brought into contact, mirror-polished, flat, clean wafers made of almost any material are attracted *via* Van der Walls forces and adhere in a rigid and permanent way. This method of direct bonding allows for new and promising designs for insulators, sensors, actuators, nonlinear optics, light-emitting diodes, etc. Solid polymers can also be brought into permanent and rigid contact to manufacture *polymer composites*. The main traditional technologies of polymer joining are mechanical fastening (bolts, rivets, fit joints) and adhesive bonding. Another technology, 'fusion bonding', presents great advantages over the previous ones, such as avoidance of high stress concentrations, reduced surface treatment, fewer inhomogeneities at the interfaces, etc. A recent book by Ageorges & Ye (2002) presents a comprehensive description of fusion bonding, defined as 'the joining of two polymer parts by the fusion and consolidation of their interface'; in particular four classes of fusion bonding are listed: 'bulk heating (co-consolidation, hot-melt adhesives, dual-resin bonding), frictional heating (spin welding, vibration welding, ultrasonic welding), electromagnetic heating (induction welding, microwave heating, dielectric heating, resistance welding), and two-stage techniques (hot plate welding, hot gas welding, radiant welding).'

For semiconductor wafer bonding, the most common and most economical combination is the silicon/silicon wafer bonding. Mozhaev *et al.* (1998) considered the theoretical implications of misorientation when two identical silicon wafers are rigidly bonded. Specifically they considered, within

the framework of anisotropic *linear elasticity*, the propagation of interface (Stoneley) waves along the bisectrix of a twist angle of misorientation. This paper deals with the propagation of waves of similar nature but within the framework of small motions superimposed on large static deformations (stress-induced anisotropy) in *nonlinear elasticity*, which is required to describe the possibly large elastic deformations of polymers (elastomers). Specifically, two half-spaces made of the same hyperelastic incompressible material are maintained in the same static state of pure homogeneous deformation and are then rigidly bonded along a common principal plane of deformation, but in such a way that the two principal axes defining this plane for one half-space do not coincide with the two counterpart principal axes for the other half-space.

The study of a superimposed infinitesimal interface wave propagating along the bisectrix of the misalignment angle provides insights into the possible ultrasonic non-destructive evaluation of the bond and of the angle of twist, and into the influence of the pre-strain on the interface stability. Indeed in general, an increasingly tensile load applied on the semi-infinite bodies leads to faster speeds for an interfacial wave whereas a compressive load slows the wave down, until eventually the buckling/bifurcation criterion is met at the critical load, where the speed is zero. Such investigations of interface waves and interfacial stability for two bonded hyperelastic pre-strained half-spaces are quite rare in the finite elasticity literature. They were initiated by Biot (1963), followed by: Chadwick & Jarvis (1979a,b), Dunwoody & Villaggio (1988), Dowaikh & Ogden (1991), Chadwick (1995), Destrade (2004a), among aothers (see Guz (2002) for pointers to the Russian and Ukrainian literature). Note, however, that in those papers, the two semi-infinite deformed bodies are always aligned so that all three principal axes of pre-deformation for the upper half-space coincide with those for the lower half-space. Moreover, the interface wave propagates along one of the common principal axes, except for Chadwick & Jarvis (1979a,b), who consider non-principal directions of propagation but for a specific simple form of strain energy function (compressible neo-Hookean materials).

Some papers have been devoted to the study of non-principal interface waves (e.g. Flavin, 1963; Willson, 1973, 1974; Chadwick & Jarvis, 1979a,b; Connor & Ogden, 1995; Rogerson & Sandiford, 1999) but explicit secular equations for tri-axially pre-strained materials have been found only in the case of neo-Hookean (compressible or incompressible) materials. This strain energy density is exceptional with respect to the propagation of inhomogeneous plane waves because for any direction of propagation in a principal plane (with associated orthogonal attenuation), the in-plane strain components always decouple from the anti-plane strain components. Section 2 presents the incremental equations of motion for non-principal interface waves in a tri-axially pre-strained material with a generic (non neo-Hookean) strain energy function, formulated as a first-order differential system for the six-component displacement–traction vector. It also presents the 'effective boundary conditions', consequences of the continuity of the mechanical displacements and tractions across the interface. Namely, these conditions are that at the boundary, either one displacement and two tractions are zero, or two displacements and one traction are zero. Mozhaev *et al.* (1998) noted this important result for bonded silicon/silicon wafers in linear anisotropic crystallography. It is further proved here that once the displacement–traction vector is normalized with respect to one of its non-zero components, then the normalized components are either real or pure imaginary (see the Appendix for details). This author recently obtained, in a quick and simple manner, some equations which are fundamental to the resolution of general interface boundary value problems (Destrade, 2003, 2004b) including Rayleigh (solid/vacuum interface), Scholte (solid/fluid interface), and Stoneley (solid/solid interface) waves; they are presented in Section 3.1. The following section brings together these fundamental equations and the effective boundary conditions for the title problem, with an explicit form of the secular equation: that is, a polynomial of which the interface wave speed is a root. Finally, Section 4 shows how numerical results can be obtained from the
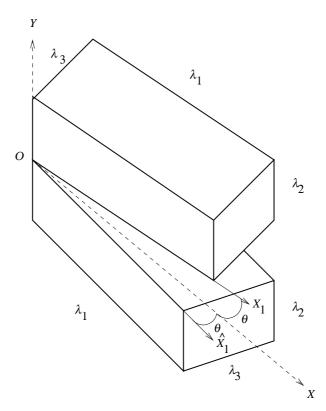
FIG. 1. Misoriented deformed cuboids at the interface.

analysis, with the case of deformed semi-infinite bodies made of Mooney–Rivlin material, a model often used to describe the behaviour of incompressible rubber in large deformations.

2. Basic equations

2.1 *Finite static pre-deformation*

Consider an infinite body made of two pre-strained hyperelastic semi-infinite bodies, rigidly bonded along a plane interface. Take the origin O of a rectangular Cartesian coordinate system ($OXYZ$) to lie in the boundary, so that the plane of separation between the half-spaces is $Y = 0$. The upper half-space $Y \geqslant 0$ is made of an incompressible isotropic hyperelastic body, with mass density ρ and strain energy function W, which has been subjected to a finite static pure homogeneous deformation with principal stretches λ_1, λ_2, λ_3 ($\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ and $\lambda_1\lambda_2\lambda_3 = 1$) say, along the principal axes $OX_1X_2X_3$ say, such that X_2 is aligned with Y but X_1 (X_3) makes an angle θ with X (Z). The lower half-space $Y \leqslant 0$ is made of the *same* body, which has been subjected to the *same* pre-deformation: that is, a pure homogeneous deformation with principal stretches λ_1, λ_2, λ_3 along the principal axes $O\hat{X}_1\hat{X}_2\hat{X}_3$ (say) where $\hat{X}_2$ is aligned with Y, but such that now $\hat{X}_1$ ($\hat{X}_3$) makes an angle $-\theta$ with X (Z). Figure 1 summarizes this set-up with the representation of two parallelepipeds, one above $Y = 0$, one below, which were unit cubes before the static deformation took place.

2.2   *Incremental equations of motion and effective boundary conditions*

Now a small-amplitude inhomogeneous plane wave is superimposed upon the primary large static deformation. The wave propagates with speed v and wave number k in the X direction and vanishes away from the interface $Y = 0$. In other words, the corresponding incremental mechanical displacement **u** is of the form

$$\mathbf{u}(x, y, z, t) = \mathbf{U}(ky)\mathrm{e}^{\mathrm{i}k(x-vt)}, \quad \mathbf{U}(\pm\infty) = \mathbf{0}, \tag{2.1}$$

where $(Oxyz) = (OXYZ)$ is the rectangular Cartesian coordinate system associated with the motion.

Similarly, the incremental nominal tractions s_{j2} ($j = 1, 2, 3$) acting on the planes $y = $ const. are of the form,

$$s_{j2}(x, y, z, t) = \mathrm{i}kt_j(ky)\mathrm{e}^{\mathrm{i}k(x-vt)}, \quad t_j(\pm\infty) = 0. \tag{2.2}$$

The following quantities allow for a compact form of the incremental equations of motion (Rogerson & Sandiford, 1999):

$$\gamma_{ij} := (\lambda_i W_i - \lambda_j W_j)\lambda_i^2/(\lambda_i^2 - \lambda_j^2) = \gamma_{ji} + \lambda_i W_i - \lambda_j W_j,$$

$$2\beta_{ij} := \lambda_i^2 W_{ii} - 2\lambda_i\lambda_j W_{ij} + \lambda_j^2 W_{jj} + 2(\lambda_i W_j - \lambda_j W_i)\lambda_i\lambda_j/(\lambda_i^2 - \lambda_j^2) = 2\beta_{ji}, \tag{2.3}$$

(where $W_i := \partial W/\partial\lambda_i$) and also

$$
\begin{aligned}
c_\theta &:= \cos\theta, & s_\theta &:= \sin\theta, \\
\eta_\theta &:= 2c_\theta^2(\beta_{12} + \gamma_{21}) + s_\theta^2\gamma_{31}, & \nu_\theta &:= c_\theta^2(\gamma_{12} - \gamma_{21}) + s_\theta^2(\gamma_{32} - \gamma_{23}), \\
\mu_\theta &:= c_\theta^2\gamma_{13} + 2s_\theta^2(\beta_{23} + \gamma_{23}), & \kappa_\theta &:= c_\theta s_\theta(\beta_{13} - \beta_{12} - \beta_{23} - \gamma_{21} - \gamma_{23}).
\end{aligned} \tag{2.4}
$$

A careful reading of the literature on incremental motions in incompressible materials (e.g. Ogden, 1984; Chadwick *et al.*, 1985; Chadwick, 1995; Rogerson & Sandiford, 1999; Destrade & Scott, 2004) and a fair amount of algebra reveal that the incremental equations of motion (in the upper half-space) can be cast in the form

$$\boldsymbol{\xi}' = \mathrm{i}\mathbf{N}\boldsymbol{\xi}, \quad \text{where} \quad \boldsymbol{\xi}(ky) = [\mathbf{U}(ky), \mathbf{t}(ky)]^{\mathrm{T}}, \tag{2.5}$$

the prime denotes differentiation with respect to ky, and **N** is the following matrix:

$$\mathbf{N} = \begin{bmatrix} 0 & -c_\theta & 0 & \frac{1}{\gamma_{21}} & 0 & 0 \\ -c_\theta & 0 & -s_\theta & 0 & 0 & 0 \\ 0 & -s_\theta & 0 & 0 & 0 & \frac{1}{\gamma_{23}} \\ X - \eta_\theta & 0 & \kappa_\theta & 0 & -c_\theta & 0 \\ 0 & X - \nu_\theta & 0 & -c_\theta & 0 & -s_\theta \\ \kappa_\theta & 0 & X - \mu_\theta & 0 & -s_\theta & 0 \end{bmatrix}, \quad X := \rho v^2. \tag{2.6}$$

The block structure of this matrix is reminiscent of that for the 'fundamental elasticity matrix' **N** of linear anisotropic elasticity (Ingebrigsten & Tonning, 1969), although its components are different. In the lower half-space the equations of motion are the same, allowing for the change in sign of s_θ and κ_θ.

At the interface, the displacement–traction vector is continuous,

$$\boldsymbol{\xi}(0^+) = \boldsymbol{\xi}(0^-) =: \boldsymbol{\xi}(0). \tag{2.7}$$

Lengthy, but straightforward to establish expressions for these quantities lead to the *effective boundary conditions* at the interface. In effect, and as displayed by Mozhaev *et al.* (1998) in the linear anisotropic elasticity case, either one displacement component and two tractions are zero at the interface, or *vice versa*. More specifically, and these details were not noted by Mozhaev *et al.* (1998), $\boldsymbol{\xi}(0)$ must be of one of the two following forms. Either

$$\boldsymbol{\xi}(0) = U_2(0)[0, 1, \mathrm{i}\alpha_2, \beta_1, 0, 0]^{\mathrm{T}}, \tag{2.8}$$

where α_2 and β_1 are real, or

$$\boldsymbol{\xi}(0) = U_1(0)[1, 0, 0, 0, \alpha_1, \mathrm{i}\beta_2]^{\mathrm{T}}, \tag{2.9}$$

where α_1 and β_2 are real. The Appendix presents the derivation of these expressions. Following Mozhaev *et al.* (1998), the interface acoustic wave satisfying (2.8) (resp. (2.9)) is called IAW1 (resp. IAW2).

3. Explicit resolution

3.1 *Fundamental equations for interface boundary value problems*

Here a quick derivation is made of some equations (Destrade, 2003) which are fundamental to the resolution of interface boundary problems involving waves or static deformations which are localized near, or equivalently vanish away from, the plane interface of a semi-infinite body. The equations are valid for unconstrained or constrained, pre-deformed nonlinearly elastic or anisotropic linearly elastic, materials. They do not rely on the Stroh (1958) formalism. All that is required for their derivation are the few manipulations, available in the literature, leading to the equations of motion and boundary conditions in the form (2.5), (2.1)$_2$, (2.2)$_2$, that is

$$\boldsymbol{\xi}' = \mathrm{i}\mathbf{N}\boldsymbol{\xi}, \quad \boldsymbol{\xi}(\infty) = \mathbf{0}. \tag{3.1}$$

In general here, $\mathbf{N}$ is a square matrix of even dimensions, $2p \times 2p$ say. Now a simple induction process (Currie, 1979; Ting, 2004) shows that $\mathbf{N}^n$, where n is any positive or negative integer, is of the following form (see (2.6) for $n = 1$):

$$\mathbf{N}^n = \begin{bmatrix} \mathbf{N_1}^{(\mathrm{n})} & \mathbf{N_2}^{(\mathrm{n})} \\ \mathbf{K}^{(\mathrm{n})} & \mathbf{N_1}^{(\mathrm{n})\mathrm{T}} \end{bmatrix}, \quad \text{with} \quad \mathbf{K}^{(\mathrm{n})} = \mathbf{K}^{(\mathrm{n})\mathrm{T}}, \quad \mathbf{N_2}^{(\mathrm{n})} = \mathbf{N_2}^{(\mathrm{n})\mathrm{T}}. \tag{3.2}$$

Consequently, pre-multiplication of $\mathbf{N}^n$ by $\hat{\mathbf{I}}$, defined as

$$\hat{\mathbf{I}} := \begin{bmatrix} \mathbf{0} & \mathbf{I_p} \\ \mathbf{I_p} & \mathbf{0} \end{bmatrix}, \quad \text{where } \mathbf{I_p} \text{ is the } p \times p \text{ identity matrix}, \tag{3.3}$$

leads to a symmetric matrix,

$$\hat{\mathbf{I}}\mathbf{N}^n = \begin{bmatrix} \mathbf{K}^{(\mathrm{n})} & \mathbf{N_1}^{(\mathrm{n})\mathrm{T}} \\ \mathbf{N_1}^{(\mathrm{n})} & \mathbf{N_2}^{(\mathrm{n})} \end{bmatrix} = (\hat{\mathbf{I}}\mathbf{N}^n)^{\mathrm{T}}. \tag{3.4}$$

Now take the scalar product of (3.1)$_1$ by $\hat{\mathbf{I}}\mathbf{N}^n\overline{\boldsymbol{\xi}}$ and add the complex conjugate quantity to obtain $\boldsymbol{\xi}' \cdot \hat{\mathbf{I}}\mathbf{N}^n\overline{\boldsymbol{\xi}} + \boldsymbol{\xi} \cdot \hat{\mathbf{I}}\mathbf{N}^n\overline{\boldsymbol{\xi}}' = 0$. Direct integration between zero and infinity yields, using (3.1)$_2$, the *fundamental equations* sought,

$$\boldsymbol{\xi}(0) \cdot \hat{\mathbf{I}}\mathbf{N}^n\overline{\boldsymbol{\xi}}(0) = 0. \tag{3.5}$$

The Cayley–Hamilton theorem states that $\mathbf{N}$ satisfies its own characteristic polynomial of degree $2p$. Hence there are only $2p - 1$ powers of $\mathbf{N}$ which are linearly independent, and so (3.5) generates at most $2p - 1$ linearly independent equations. For instance, in our case $\mathbf{N}$ is a 6×6 matrix, leading to only five linearly independent equations. These are nevertheless enough to deduce the secular equation explicitly, as is now seen.

3.2   *Secular equation*

The fundamental equations (3.5) now prove useful in the resolution of the paper's problem. The structure of $\hat{\mathbf{I}}\mathbf{N}^n$ depends on the parity of n. For instance,

$$\hat{\mathbf{I}}\mathbf{N}^n = \begin{bmatrix} 0 & N_{42}^n & 0 & N_{11}^n & 0 & N_{31}^n \\ N_{42}^n & 0 & N_{53}^n & 0 & N_{22}^n & 0 \\ 0 & N_{53}^n & 0 & N_{13}^n & 0 & N_{33}^n \\ N_{11}^n & 0 & N_{13}^n & 0 & N_{15}^n & 0 \\ 0 & N_{22}^n & 0 & N_{15}^n & 0 & N_{26}^n \\ N_{31}^n & 0 & N_{33}^n & 0 & N_{26}^n & 0 \end{bmatrix}, \quad \text{for } n = -2, 2, \tag{3.6}$$

where $N_{ij}^n := (\mathbf{N}^n)_{ij}$; thus for example, $N_{42}^2 := (\mathbf{N}^2)_{42} = N_{4k}N_{k2}$. The forms (2.8) and (2.9) for $\boldsymbol{\xi}(0)$ lead to two trivial identities when (3.5) are written at $n = -2, 2$. On the other hand,

$$\hat{\mathbf{I}}\mathbf{N}^n = \begin{bmatrix} N_{41}^n & 0 & N_{43}^n & 0 & N_{21}^n & 0 \\ 0 & N_{52}^n & 0 & N_{12}^n & 0 & N_{32}^n \\ N_{43}^n & & N_{63}^n & 0 & N_{23}^n & 0 \\ 0 & N_{12}^n & 0 & N_{14}^n & 0 & N_{16}^n \\ N_{21}^n & & N_{23}^n & 0 & N_{25}^n & 0 \\ 0 & N_{32}^n & 0 & N_{16}^n & 0 & N_{36}^n \end{bmatrix}, \quad \text{for } n = -1, 1, 3, \tag{3.7}$$

and so the fundamental equations (3.5) written for IAW1, that is $\boldsymbol{\xi}(0)$ given by (2.8), and for $n = -1, 1, 3$, yield the following system:

$$\begin{bmatrix} N_{12}^{-1} & N_{14}^{-1} & N_{63}^{-1} \\ N_{12}^{1} & N_{14}^{1} & N_{63}^{1} \\ N_{12}^{3} & N_{14}^{3} & N_{63}^{3} \end{bmatrix} \begin{bmatrix} 2\beta_1 \\ \beta_1^2 \\ \alpha_2^2 \end{bmatrix} = \begin{bmatrix} -N_{52}^{-1} \\ -N_{52}^{1} \\ -N_{52}^{3} \end{bmatrix}. \tag{3.8}$$

When the fundamental equations (3.5) are written for IAW2, that is $\boldsymbol{\xi}(0)$ given by (2.9), and for $n = -1, 1, 3$, they yield the system

$$\begin{bmatrix} N_{21}^{-1} & N_{25}^{-1} & N_{36}^{-1} \\ N_{21}^{1} & N_{25}^{1} & N_{36}^{1} \\ N_{21}^{3} & N_{25}^{3} & N_{36}^{3} \end{bmatrix} \begin{bmatrix} 2\alpha_1 \\ \alpha_1^2 \\ \beta_2^2 \end{bmatrix} = \begin{bmatrix} -N_{41}^{-1} \\ -N_{41}^{1} \\ -N_{41}^{3} \end{bmatrix}. \tag{3.9}$$

These equations are of the type $M_{ik}b_k = a_i$ where here $\mathbf{M}$ is the 3×3 matrix on the left-hand side of (3.8) for IAW1 and of (3.9) for IAW2, $\mathbf{a}$ is the vector on the right-hand side, and $\mathbf{b} = [2\beta_1, \beta_1^2, \alpha_2^2]^{\mathrm{T}}$, $[2\alpha_1, \alpha_1^2, \beta_2^2]^{\mathrm{T}}$, for IAW1 and for IAW2, respectively. By Cramer's rule, the unique solution is $b_k = \Delta_k/\Delta$, where $\Delta = \det \mathbf{M}$, and Δ_k is the determinant of the matrix obtained from $\mathbf{M}$ by replacing the kth vector column with $\mathbf{a}$. However, two components of $\mathbf{b}$ are related one to another. Specifically,
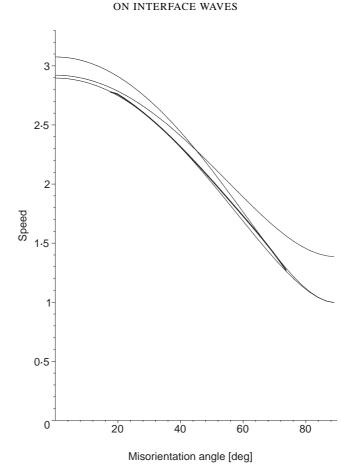
FIG. 2. Mooney–Rivlin bimaterial subject to the triaxial pre-stretch $\lambda_1^2 = 3{\cdot}695$, $\lambda_2^2 = 0{\cdot}7$, $\lambda_3^2 = 0{\cdot}387$: plot of the Stoneley wave speed with θ (thick curve), bounded above by the shear wave speeds (crossing at $\theta = 45^{\circ}$) and below by the Rayleigh wave speed.

$b_1^2 = 4b_2$, which is the *explicit secular equation for interface waves propagating along the bisectrix of the misorientation angle for two identical, rigidly bonded, pre-deformed, hyperelastic half-spaces,*

$$\Delta_1^2 - 4\Delta\Delta_2 = 0. \tag{3.10}$$

4. Numerical results for rubberlike materials

The classical Mooney–Rivlin model has been used extensively to model the behaviour of incompressible rubberlike materials in large deformations. Its strain energy function W is given by

$$2W = C(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + D(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3), \tag{4.1}$$

where C and D are constant material parameters. In that case, the γ_{ij} and β_{ij} defined in (2.3) reduce to ($i \neq j \neq k \neq i$)

$$\gamma_{ij} = \lambda_i^2(C + D\lambda_k^2), \quad 2\beta_{ij} = (\lambda_i^2 + \lambda_j^2)(C + D\lambda_k^2) = \gamma_{ij} + \gamma_{ji}. \tag{4.2}$$
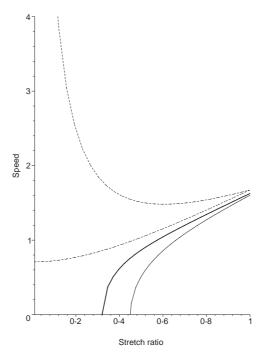
FIG. 3. Deformed Mooney–Rivlin bimaterial in compressive plane strain at $\theta = 30^o$: variations with λ of the shear (dashed curves), Stoneley (thick curve), and Rayleigh (lower curve) wave speeds.

Now the peculiarities of interface acoustic waves in deformed hyperelastic materials are highlighted. In particular, and in contrast to the corresponding situation in linear anisotropic elasticity, it is seen that there exist certain ranges of misorientation and certain ranges of pre-stretch ratios for the existence of an IAW. Also, interface instability may arise at certain compressive critical ratios.

4.1 Mooney–Rivlin material in tri-axial strain

First, in order to make the connection with results obtained by Rogerson & Sandiford (1999) about non-principal surface waves, the material constants C, D, and the stretch ratios λ_i^2 are fixed at the following values:

$$C = 2 \cdot 0, \quad D = 0 \cdot 8, \quad \lambda_1^2 = 3 \cdot 695, \quad \lambda_2^2 = 0 \cdot 7, \quad \lambda_3^2 = 0 \cdot 387. \tag{4.3}$$

The secular equation (3.10) is a polynomial of degree 10 in $X = \rho v^2$ for IAW1, of degree 6 for IAW2. Out of the 16 possible roots, only one leads to an interface wave of the Stoneley type satisfying the known conditions that its speed is bounded above by the speed of the slowest homogeneous bulk wave and below by the speed of the Rayleigh surface wave associated with either half-space (Barnett *et al.*, 1985). Moreover, this wave, of IAW1 type, satisfies these requirements only within a limited range of misorientation angle, approximatively $16 \cdot 7^o < \theta < 73 \cdot 6^o$. This situation is in sharp contrast with the case of silicon/silicon wafers in linear anisotropic elasticity (Mozhaev *et al.*, 1998) where the IAW1 was found to exist for all θ.

Figure 2 depicts the variations of the relevant root to the secular equation, scaled as $\sqrt{X} = \sqrt{\rho v^2}$,

with θ (thick curve). The speeds of two homogeneous shear waves (roots of det $\mathbf{N} = 0$) are represented by the two thin curves above, and the speed of the Rayleigh wave by the thin curve below.

4.2 *Mooney–Rivlin material in compressive plane strain*

Next, the half-spaces are assumed to have been pre-deformed in such a way that they were not allowed to expand in the Z-direction ($\lambda_3 = 1$). For this example, the different parameters take the following numerical and algebraic values:

$$C = 2{\cdot}0, \quad D = 0{\cdot}8, \quad \theta = 30^{\text{o}}, \quad \lambda_1 = \lambda, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1. \tag{4.4}$$

Figure 3 shows the variations of the speeds as functions of λ in compression ($\lambda < 1$). The two upper dashed curves represent the speeds of homogeneous shear waves; the intermediate thick curve, the speed of the Stoneley wave (type: IAW1); the bottom thin curve, the speed of the Rayleigh wave associated with either deformed half-space. Again, the situation is different from that encountered in linear anisotropic elasticity with silicon/silicon wafers (Mozhaev *et al.*, 1998). In particular, the Stoneley wave exists only for stretch ratios greater than $0{\cdot}321$ and the Rayleigh wave for stretch ratios greater than $0{\cdot}451$. In between these two critical stretches, there is a range where the Stoneley wave exists but not the Rayleigh wave. At the critical stretch, instability might occur; hence is presented an example where two bonded deformed half-spaces exhibit interfacial stability at some compressive stretch ratios for which the separated half-spaces are unstable (at least in the linearized theory).

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Appendix: effective boundary conditions

Here two main results about the boundary conditions of the paper's problem are established. The first result is that at the interface, either one displacement component and two tractions are zero, or *vice versa*. Mozhaev *et al.* (1998) found this result for the corresponding problem in anisotropic linear elasticity; because of space limitations they only sketched the proof. The second result is that once the displacement–traction vector at the interface is normalized with respect to one of the three non-zero components, the two others are either purely real or purely imaginary quantities. This Appendix is a generalization of results obtained by Ting (2004) for surface (Rayleigh) waves and by Destrade (2003) for 2-partial interface (Stoneley) waves.

Consider, for the *upper half-space*, a combination of inhomogeneous plane waves of the form

$$\boldsymbol{\xi} = \boldsymbol{\xi}^i e^{ik(x+p_i y - vt)}, \tag{A1}$$

where $\boldsymbol{\xi}^i$ is a constant vector and $\Im(p_i) > 0$ to ensure decay of the wave with distance from the interface. Substitution of this form of solution into the equations of motion (2.5) shows that the p_i are roots of the bicubic: $\det(\mathbf{N} - p\mathbf{1}) = 0$, or (Rogerson & Sandiford, 1999)

$$\gamma_{21}\gamma_{23}p^6 - [(\gamma_{21} + \gamma_{23})X - c_1]p^4 + (X^2 - c_2 X + c_3)p^2 + (X - c_4)(X - c_5) = 0, \tag{A2}$$

with

$$
\begin{aligned}
c_1 &= (\gamma_{21}\gamma_{13} + 2\beta_{12}\gamma_{23})c_\theta^2 + (\gamma_{23}\gamma_{31} + 2\beta_{23}\gamma_{21})s_\theta^2, \\
c_2 &= (\gamma_{23} + \gamma_{13} + 2\beta_{12})c_\theta^2 + (\gamma_{21} + \gamma_{31} + 2\beta_{23})s_\theta^2, \\
c_3 &= (\gamma_{12}\gamma_{23} + 2\beta_{12}\gamma_{13})c_\theta^4 + (\gamma_{21}\gamma_{32} + 2\beta_{23}\gamma_{31})s_\theta^4 \\
&\quad + [\gamma_{12}\gamma_{21} + \gamma_{13}\gamma_{31} + \gamma_{23}\gamma_{32} - (\beta_{13} - \beta_{12} - \beta_{23})^2 + 4\beta_{12}\beta_{23}]c_\theta^2 s_\theta^2, \\
c_4 &= \gamma_{12}c_\theta^2 + \gamma_{32}s_\theta^2, \\
c_5 &= \gamma_{13}c_\theta^4 + 2\beta_{13}c_\theta^2 s_\theta^2 + \gamma_{31}s_\theta^4.
\end{aligned}
\tag{A3}
$$

Let p_1, p_2, p_3 be the three roots with positive imaginary part. Then the wave combination is

$$\boldsymbol{\xi} = \gamma_1 \boldsymbol{\xi}^1 e^{ik(x+p_1 y - vt)} + \gamma_2 \boldsymbol{\xi}^2 e^{ik(x+p_2 y - vt)} + \gamma_3 \boldsymbol{\xi}^3 e^{ik(x+p_3 y - vt)}, \quad (y \geqslant 0), \tag{A4}$$

where the γ_i are constant numbers.

In the *lower half-space*, the equations of motion are of the same form as in the upper half-space with θ changed to $-\theta$. Hence only s_θ and κ change signs in the components of $\mathbf{N}$ whilst the coefficients of the associated characteristic polynomial remain unchanged. Consequently, a suitable combination of inhomogeneous plane waves is here

$$\hat{\boldsymbol{\xi}} = \hat{\gamma}_1 \hat{\boldsymbol{\xi}}^1 e^{ik(x-p_1 y - vt)} + \hat{\gamma}_2 \hat{\boldsymbol{\xi}}^2 e^{ik(x-p_2 y - vt)} + \hat{\gamma}_3 \hat{\boldsymbol{\xi}}^3 e^{ik(x-p_3 y - vt)}, \quad (y \leqslant 0), \tag{A5}$$

where the $\hat{\gamma}_i$ are constant numbers and the $\hat{\boldsymbol{\xi}}^i$ are constant vectors.

Now two possibilities arise for the roots p_1, p_2, p_3 of (A2). Either

$$\text{(I)} \ p_i = iq_i \ (q_i > 0), \quad \text{or} \quad \text{(II)} \ p_1 = -\overline{p_2}, \quad p_3 = iq_3 \ (q_3 > 0). \tag{A6}$$

In case (I) it is a straightforward matter to show, using the adjoint of $(\mathbf{N} - p\mathbf{1})$, that the $\boldsymbol{\xi}^i$ (upper half-space) can be written in the form

$$\boldsymbol{\xi}^i = [a_i, ib_i, c_i, id_i, e_i, if_i]^{\mathrm{T}}, \tag{A7}$$

say, where $a_i, b_i, c_i, d_i, e_i, f_i$ ($i = 1, 2, 3$) are *real* numbers. Furthermore, the vectors $\hat{\boldsymbol{\xi}}^i$ (lower half-space, where θ is changed to $-\theta$) are then in the form

$$\hat{\boldsymbol{\xi}}^i = [-a_i, ib_i, c_i, id_i, -e_i, -if_i]^{\mathrm{T}}. \tag{A8}$$

At the interface $y = 0$ the displacement–traction vector is continuous, $\boldsymbol{\xi}(x, 0, t) = \hat{\boldsymbol{\xi}}(x, 0, t)$, or

$$\gamma_1 \boldsymbol{\xi}^1 + \gamma_2 \boldsymbol{\xi}^2 + \gamma_3 \boldsymbol{\xi}^3 = \hat{\gamma}_1 \hat{\boldsymbol{\xi}}^1 + \hat{\gamma}_2 \hat{\boldsymbol{\xi}}^2 + \hat{\gamma}_3 \hat{\boldsymbol{\xi}}^3. \tag{A9}$$

These six equations are recast as

$$\mathbf{A}(\boldsymbol{\gamma} + \hat{\boldsymbol{\gamma}}) = \mathbf{0}, \quad \mathbf{B}(\boldsymbol{\gamma} - \hat{\boldsymbol{\gamma}}) = \mathbf{0}, \tag{A10}$$

where

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ e_1 & e_2 & e_3 \\ if_1 & if_2 & if_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} ib_1 & ib_2 & ib_3 \\ c_1 & c_2 & c_3 \\ id_1 & id_2 & id_3 \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}, \quad \hat{\boldsymbol{\gamma}} = \begin{bmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \\ \hat{\gamma}_3 \end{bmatrix}. \tag{A11}$$

For non-trivial solutions to exist, either (a) $\det \mathbf{A} = 0$ or (b) $\det \mathbf{B} = 0$. In case (a), equation (A10)$_2$ leads to $\boldsymbol{\gamma} = \hat{\boldsymbol{\gamma}}$. Then (A10)$_1$ reads

$$\mathbf{A}\boldsymbol{\gamma} = \mathbf{0}. \tag{A12}$$

Owing to the form (A11)$_1$ of $\mathbf{A}$, this condition is satisfied when the components of $\boldsymbol{\gamma}$ are all real. In conclusion, $\boldsymbol{\xi}(0)$ now reads

$$\boldsymbol{\xi}(0) = \gamma_1 \begin{bmatrix} 0 \\ ib_1 \\ c_1 \\ id_1 \\ 0 \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ ib_2 \\ c_2 \\ id_2 \\ 0 \\ 0 \end{bmatrix} + \gamma_3 \begin{bmatrix} 0 \\ ib_3 \\ c_3 \\ id_3 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma_i \text{ real}, \tag{A13}$$

that is $\boldsymbol{\xi}(0)$ is of the form (2.8). A similar procedure shows that in case (b), $\boldsymbol{\xi}(0)$ is of the form (2.9).

In case (II) the $\boldsymbol{\xi}^i$ (upper half-space) can be written in the form

$$\boldsymbol{\xi}^1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \\ f_1 \end{bmatrix}, \quad \boldsymbol{\xi}^2 = \begin{bmatrix} \overline{a_1} \\ -\overline{b_1} \\ \overline{c_1} \\ -\overline{d_1} \\ \overline{e_1} \\ -\overline{f_1} \end{bmatrix}, \quad \boldsymbol{\xi}^3 = \begin{bmatrix} a_3 \\ ib_3 \\ c_3 \\ id_3 \\ e_3 \\ if_3 \end{bmatrix}, \tag{A14}$$

say, where $a_1, b_1, c_1, d_1, e_1, f_1$ are *complex* and $a_3, b_3, c_3, d_3, e_3, f_3$ are *real*. The vectors $\hat{\boldsymbol{\xi}}^i$ (lower half-space) are then in the form

$$\hat{\boldsymbol{\xi}}^1 = \begin{bmatrix} -a_1 \\ b_1 \\ c_1 \\ d_1 \\ -e_1 \\ -f_1 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}^2 = \begin{bmatrix} -\overline{a_1} \\ -\overline{b_1} \\ \overline{c_1} \\ -\overline{d_1} \\ -\overline{e_1} \\ \overline{f_1} \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}^3 = \begin{bmatrix} -a_3 \\ ib_3 \\ c_3 \\ id_3 \\ -e_3 \\ -if_3 \end{bmatrix}. \tag{A15}$$

The continuity of the displacement–traction vector at $y = 0$ can again be written in the form (A11) where now

$$\mathbf{A} = \begin{bmatrix} a_1 & \overline{a_1} & a_3 \\ e_1 & \overline{e_1} & e_3 \\ f_1 & -\overline{f_1} & if_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 & -\overline{b_1} & ib_3 \\ c_1 & \overline{c_1} & c_3 \\ d_1 & -\overline{d_1} & d_3 \end{bmatrix}. \tag{A16}$$

Again case (a) det $\mathbf{A} = 0$ or case (b) det $\mathbf{B} = 0$ arises. In case (a), $\boldsymbol{\gamma} = \hat{\boldsymbol{\gamma}}$ and so $\mathbf{A}\boldsymbol{\gamma} = \mathbf{0}$. Owing to the form (A16)$_1$ of $\mathbf{A}$, this condition is satisfied when $\boldsymbol{\gamma}$ is of the form $\boldsymbol{\gamma} = [\gamma_1, \overline{\gamma_1}, \gamma_3]^{\mathrm{T}}$, where γ_1 is complex and γ_3 is real. In conclusion, $\boldsymbol{\xi}(0)$ now reads

$$\boldsymbol{\xi}(0) = \gamma_1 \begin{bmatrix} 0 \\ b_1 \\ c_1 \\ d_1 \\ 0 \\ 0 \end{bmatrix} + \overline{\gamma_1} \begin{bmatrix} 0 \\ -\overline{b_1} \\ \overline{c_1} \\ -\overline{d_1} \\ 0 \\ 0 \end{bmatrix} + \gamma_3 \begin{bmatrix} 0 \\ ib_3 \\ c_3 \\ id_3 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma_3 \text{ real}, \tag{A17}$$

which means that, once normalized, $\boldsymbol{\xi}(0)$ is of the form (2.8). A similar procedure shows that in case (b), $\boldsymbol{\xi}(0)$ is of the form (2.9).