Weierstrass's criterion and compact solitary waves

Michel Destrade*
Institut Jean Le Rond d'Alembert, UMR 7190, CNRS, Université Pierre et Marie Curie, 4 place Jussieu, case 162, 75252 Paris Cedex 05, France

Giuseppe Gaeta†
Dipartimento di Matematica, Università di Milano, Via Saldini, 50, 20133 Milano, Italy

Giuseppe Saccomandi‡
Sezione di Ingegneria Industriale, Dipartimento di Ingegneria dell'Innovazione, Università degli Studi di Lecce, 73100 Lecce, Italy
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Weierstrass's theory is a standard qualitative tool for single degree of freedom equations, used in classical mechanics and in many textbooks. In this Brief Report we show how a simple generalization of this tool makes it possible to identify some differential equations for which compact and even semicompact traveling solitary waves exist. In the framework of continuum mechanics, these differential equations correspond to bulk shear waves for a special class of constitutive laws.

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I. INTRODUCTION

A *compact wave* is a robust solitary wave with a compact support, outside of which it vanishes identically. A *compacton* is to a compact wave what a soliton is to a solitary wave with an infinite support; that is, a compacton is a compact wave that preserves its shape after colliding with another compact wave. Rosenau and Hyman [1] introduced these concepts over a decade ago and a substantial number of differential equations supporting compact waves has since been identified and studied.

From the mathematical point of view, the emergence of such solutions is related to the degeneracy of the differential equations of motion at certain points, and to the corresponding failure of the uniqueness theorem at these. Indeed, compact waves are a patchwork made by pasting together at degenerate points the different possible solutions (unique in between degenerate points) and hence by patching together nonunique solutions; it follows that they are not analytic solutions, and in this respect they are substantially different from standard soliton solutions.

Compact waves are weak solutions of differential equations, which are continuous—in contrast to shock waves—but have discontinuous derivatives, similarly to acceleration waves. A simple introduction to compact waves and to compactons can be found in a recent article by Rosenau [2].

The main objection facing compact waves is that the link between an *adequate* partial differential equation and a *constitutive* law is often tenuous. Indeed, the generic affirmation that compact waves emerge from a balance between higher nonlinearity and nonlinear dispersion remains vague and esoteric if it is not supported by a clear and rigorous mechanical derivation of the right equations.

Destrade and Saccomandi [3,4] recently proposed a general theory of dispersive nonlinear acoustics and showed how it is possible to derive exact equations governing the propagation of compact shear waves in solids with an inherent characteristic length. By applying standard asymptotic procedures to these exact equations, it is then possible to justify evolutions equations which are similar to the compacton factory known as the $K(m,n)$ Korteweg–de Vries equation.

From the mechanical point of view, these results were obtained by a careful modeling of the dispersive part of the Cauchy stress tensor. The corresponding constitutive equations unify and explain in depth several theories of weakly nonlocal continuum mechanics, such as the Lagrangian Averaged Navier-Stokes-α theory of turbulence, or the Rubin *et al.* [5] theory of inherent characteristic length.

From the mathematical point of view, the emergence of compact waves is investigated by a simple modification of Weierstrass's theory (a useful qualitative tool for single degree of freedom equations in classical mechanics).

In this Brief Report we show how another class of constitutive assumptions can cause the emergence of compact waves and even of semicompact waves, which are traveling solitary waves with a semi-infinite support.

II. A GENERALIZATION OF WEIERSTRASS'S DISCUSSION

One of the most elegant and powerful tools for the qualitative analysis of one-dimensional Lagrangian conservative motions is Weierstrass's theory. When an integral of energy (in a generalized sense) exists, Weierstrass's theory allows us to understand whether the motion of our Lagrangian system is periodic or nonperiodic, simply by looking at the roots of the potential function.

Let us imagine that our natural Lagrangian system can be described by a single holonomic parameter q, say. Furthermore, suppose that it is time independent. The energy integral of such a system is

*Electronic address: destrade@lmm.jussieu.fr
†Electronic address: Giuseppe.Gaeta@mat.unimi.it
‡Electronic address: giuseppe.saccomandi@unile.it

$$T(q,\dot{q}) - U(q) = E, \tag{1}$$

where T is the kinetic energy, U is minus the potential energy (note the unconventional choice of sign), and the constant E is determined by initial conditions.

Recall that, for natural Lagrangian systems, the most general form of the kinetic energy is $T=a(q)\dot{q}^2/2$, where $a=a(q)$ is a positive function of q only; also, $T=0$ only if $\dot{q}=0$. We will thus assume that (1) is in the form

$$\frac{1}{2}a(q)\dot{q}^2 - U(q) = E. \tag{2}$$

It follows that our motion is confined to those configurations where $U(q)+E \geqslant 0$. The special configurations $\bar{q}$ (say) where $U(\bar{q})+E=0$ and $U'(\bar{q}) \neq 0$ are called *barriers* because they cannot be crossed by the motion of $q(t)$: they split the range of possible values for q into allowed and prohibited intervals. Points with $U(\bar{q})+E=0$ and $U'(\bar{q})=0$ are *soft barriers* and separate two allowed intervals (see below).

Solving first (2) for $\dot{q}$, and then differentiating with respect to t, we find in turn

$$\dot{q}^2 = 2\frac{U(q)+E}{a(q)}, \tag{3a}$$

$$\ddot{q} = \frac{1}{a(q)}U'(q) - \frac{a'(q)}{a^2(q)}[U(q)+E]. \tag{3b}$$

Hence at a barrier $\bar{q}$ which is a simple root of $U+E=0$ [that is, such that $U(\bar{q})+E=0$ and $U'(\bar{q}) \neq 0$], we have $\ddot{q} \neq 0$. Such a barrier is called an *inversion point*, because the motion reverses its course after reaching it.

We now separate the variables in (3a) and integrate to find

$$t = \pm \int \sqrt{\frac{a(q)}{2[U(q)+E]}}dq. \tag{4}$$

At a (soft) barrier which is a double root of $U+E=0$ [that is, such that $U(\bar{q})+E=U'(\bar{q})=0$], the integral diverges. Thus a soft barrier is also called an *asymptotic point* because it takes an infinite time to reach it.

III. WAVE PROPAGATION

Now we turn to wave propagation. Consider the case of semilinear wave equations in the unknown field $u=u(x,t)$,

$$u_{tt} - c^2 u_{xx} = F(u), \tag{5}$$

where c is a constant and F a nonlinear function of u. Then for traveling waves of speed v, i.e., for u in the form

$$u(x,t) = \varphi(z), \quad z := x - vt, \tag{6}$$

we obtain the second-order differential equation

$$(v^2 - c^2)\varphi'' - F(\varphi) = 0. \tag{7}$$

Multiplying by φ' and integrating, we find an equation of the same form as Eq. (3a), where a is the constant v^2-c^2 and U is the antiderivative of F; note that E is now related to the integration constant.

Weierstrass's theory tells us that a periodic wave corresponds to the existence of two consecutive inversion points; that a pulse solitary wave with infinite tails corresponds to the existense of an asymptotic point followed by an inversion point; and that a kink solitary wave with infinite tails corresponds to two consecutive asymptotic points; see Peyrard and Dauxois [6] or Kichenassamy and Olver [7] for similar discussions.

Consider the case of the following fully nonlinear wave equations,

$$u_{tt} - c^2 u_{xx} - c_{\mathrm{NL}}^2 (u_x^3)_x = F(u), \tag{8}$$

where c_{NL} is a constant. The traveling wave reduction (6) yields

$$(v^2 - c^2)\varphi'' - c_{\mathrm{NL}}^2[(\varphi')^3]' = F(\varphi). \tag{9}$$

Here Weierstrass's discussion must be modified, sometimes to dramatic effect.

Take, for instance, the degenerate case $v^2=c^2$. In that case, the first integral of (9) is

$$(\varphi')^4 = 2\frac{U(\varphi)+E}{a}, \tag{10}$$

where $U(\varphi) \equiv -\int F$, E is a constant of integration, and $a=3c_{\mathrm{NL}}^2/2$.

We can still conduct an analysis in the manner of Weierstrass, but we find that the fourth-order power above introduces some new features, not present in mechanical conservative Lagrangian systems. Indeed, now the barriers $\bar{\varphi}$ are attainable not only when they are simple roots of $U+E=0$, but also when they are double roots. This is the case because the analog to (4) is here

$$z = \pm \int \sqrt[4]{\frac{a}{2[U(\varphi)+E]}}d\varphi, \tag{11}$$

and the integral converges for double roots.

Saccomandi [8] gives a detailed discussion on this possibility and explains its consequences by the failure of the Lipschitz condition, leading to a possible lack of uniqueness, and eventually to the possibility of compact waves; see Destrade and Saccomandi [3,4] for examples.

In the present note we study yet another possibility for Weierstrass's theory, touched upon by Ferrari and Moscatelli [9], Rosenau [10], and Gaeta *et al.* [11]; namely, the case of barriers which are roots of noninteger order to the equation $U+E=0$.

For a first glimpse at what may happen in this case, we consider in turn two examples with roots of fractionary order.

Example 1. In the first example, we take $E=0$, $a=2$, and

$$U(q) = \sqrt{q}(1-\sqrt{q}). \tag{12}$$

Here we record two barriers: $\bar{q}_1=0$ and $\bar{q}_2=1$. The integral in (4) yields

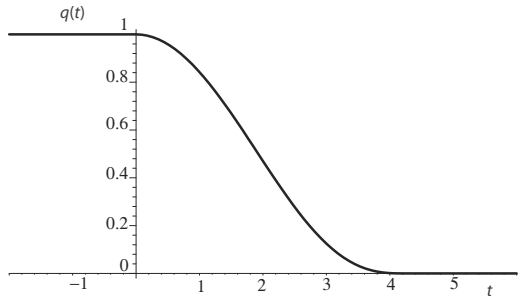

FIG. 1. Example 2. Compact solitary kink wave, patched together using fractional roots in Weierstrass's discussion, with a potential of the form $U(q)=q^{4/3}(1-q^{1/3})$.

$$t = -2\sqrt{\sqrt{q}-q} + \arcsin(2\sqrt{q}-1). \tag{13}$$

Clearly, both barriers can be reached in finite times: with our choice of the integration constant, these are $t=-\pi/2$ for $\bar{q}_1$ and $t=0$ for $\bar{q}_2$. From (3) the acceleration is

$$\ddot{q} = \frac{1-2\sqrt{q}}{2\sqrt{q}}. \tag{14}$$

At $\bar{q}_2=1$ the acceleration is not zero, and the motion reverses; at $\bar{q}_1=0$, however, the acceleration blows up.

Example 2. In the second example, we take $E=0$, $a=2$, and

$$U(q) = q^{4/3}(1-q^{1/3}). \tag{15}$$

Here we record two barriers as well, again $\bar{q}_1=0$ and $\bar{q}_2=1$. We perform the integral in (4) for $t \in [-3\sqrt{2}, 3\sqrt{2}]$ and solve it explicitly for q as

$$q(t) = \left(1 - \frac{t^2}{18}\right)^3. \tag{16}$$

Clearly, again, the barriers are reached in finite times: $\bar{q}_1=0$ at $t=\pm 3\sqrt{2}$ and $\bar{q}_2=1$ at $t=0$. We also find that the acceleration is given by

$$\ddot{q} = \frac{1}{3}q^{1/3}(4-5q^{1/3}), \tag{17}$$

making it clear that the barrier $\bar{q}_2=1$ is a configuration associated with a finite force, whereas the barrier $\bar{q}_1=0$ is associated with an equilibrium. Furthermore, note that the right-hand side of (17) does not satisfy the Lipschitz condition at $q=\bar{q}_1=0$. This allows nonuniqueness of the solution, and in fact we can patch together a compact solitary kink: this is equal to 1 for $t<0$, then for $0<t<3\sqrt{2}$ it is accelerated to negative velocity and decreases to reach zero at time $t=3\sqrt{2}$, and then stays there for $t>3\sqrt{2}$; see Fig. 1.

These two examples highlight the complexity and the richness of the situations arising when considering noninteger barriers.

Let us now turn to a more general (nonfractional) case. We take $E=0$, $a=2$, and

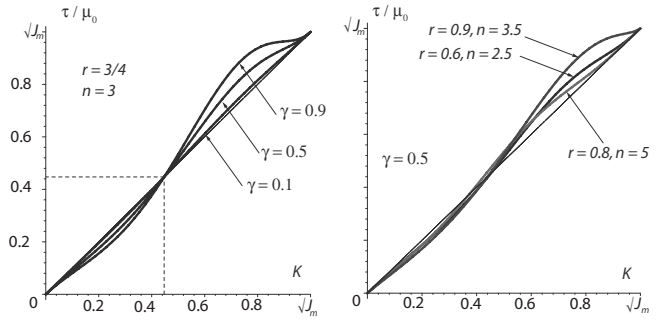

FIG. 2. Shear stress response obtained by varying the constitutive parameters in (27). The plots cross the linear stress shear response (thin straight line) at $K=0$, $\sqrt{rJ_m/(r+n)}$ (indicated by dashed lines on left graph), and $\sqrt{J_m}$.

$$U(q) = q^{2r}V(q^2), \tag{18}$$

where V, and thus U, has a barrier $\bar{q}>0$ $[V(\bar{q}^2)=0]$. Then by Eq. (3b), the acceleration is

$$\ddot{q} = rq^{2r-1}V(q^2) + q^{2r+1}V'(q^2). \tag{19}$$

We should consider several cases, depending on the value of $r>0$.

If $2r<1$, then q blows up at the barrier $q\to 0$, similarly to the first example above. For this case we conclude that sublinear $(2r<1)$ roots correspond to singular points which cannot be reached in any way.

If $2r=1$ or $2r=2$, we recover the already discussed cases of simple and double roots.

If $2r\geqslant 2$, the integral in (4) diverges and the barrier at zero cannot be reached in a finite time—it is an asymptotic point.

If $1<2r<2$, however, the integral is finite and the barrier zero is reached in a finite time. Also, by (19), $\ddot{q}=0$ at that barrier, and we thus arrive at an equilibrium configuration with null velocity, a situation that gives rise to a predicament: will the motion settle on this equilibrium configuration *ad infinitum* or will it reverse its course?

In fact, the Cauchy problem is ill posed here, and the accompanying lack of uniqueness gives us some latitude to patch together compact waves (similarly to the second example above) or semicompact waves, as is seen in the next section.

IV. SEMICOMPACT SHEAR STRAIN WAVES

Destrade and Saccomandi [3,4] show that the motion of transverse strain waves in nonlinear dispersive solids is governed by the equation

$$(\mu W)_{xx} + (\alpha W_{tt})_{xx} = \rho W_{tt}, \tag{20}$$

where x is the direction of propagation, W is the transverse strain, and ρ is the mass density. The constitutive parameters are $\mu=\mu(W^2)$, the generalized shear modulus of nonlinear elasticity, and α, the dispersion parameter of Rubin *et al.* [5]. For simplicity here, α is taken constant and the strain wave is linearly polarized, traveling with speed v; see (6). Then $W=W(x-vt)$ and (20) leads to

$$\mu W + \alpha v^2 W'' = \rho v^2 W. \tag{21}$$

Next we write μ in the form

$$\mu(W^2) = \mu_0 - (\alpha\mu_0/\rho)[rW^{2r-1}V(W^2) + W^{2r+1}V'(W^2)], \tag{22}$$

where μ_0 is the ground state shear modulus, $r>1/2$, and V is an as yet arbitrary function. For bulk waves, v is arbitrary and here we fix it at the sonic speed $v \equiv \sqrt{\mu_0/\rho}$. Then (21) is exactly of the same form as (19).

We are thus entitled to consider the possibility of barriers of noninteger order for the wave. For instance, take V in the form

$$V(W^2) = \gamma\frac{\rho}{\alpha}(J_m - W^2)^n, \tag{23}$$

where $\gamma>0$, $J_m>0$, and $n>1$ are constants. Integrating (22) with this choice gives

$$W'^2 = \gamma\frac{\rho}{\alpha}W^{2r}(J_m - W^2)^n. \tag{24}$$

Then the following change of variables and rescaling of the function

$$\xi \equiv \left[\gamma\frac{\rho}{\alpha}J_m^{r+n-1}\right]^{1/2} z, \quad \omega(\xi) \equiv W[z(\xi)]/\sqrt{J_m} \tag{25}$$

give the following nondimensional version of the governing equation:

$$\dot{\omega}^2 = \omega^{2r}(1-\omega^2)^n. \tag{26}$$

Now we briefly discuss whether (22) and (23) constitutes a reasonable shear response for a nonlinear solid. We recall that the shear stress τ (say) necessary to maintain a solid in a static state of finite shear with amount of shear K (say) is $\tau = \mu(K^2)K$, given here by

$$\frac{\tau}{\mu_0} = K - \gamma[rK^{2r}(J_m - K^2)^n - nK^{2r+2}(J_m - K^2)^{n-1}]. \tag{27}$$

First we see that K must not be allowed to go too far beyond $\sqrt{J_m}$ for τ to remain positive. In practice, this means that for a given material we must fix J_m beyond the maximal shear allowed before its rupture. Note that the actual value of J_m has no bearing on the existence and characteristics of the shear wave because it does not appear in (26).

Second, we remark that the graph of $\tau(K)$ and the graph of $\mu_0 K$ (corresponding to a material with a linear shear response) cross—independent of γ—for $K=0$, $\sqrt{rJ_m/(r+n)}$, and $\sqrt{J_m}$. The slopes of the $\tau(K)$ graph at $K=0$ and at $K=\sqrt{J_m}$ are defined (and equal to μ_0) when

$$2r > 1 \quad \text{and} \quad n > 2. \tag{28}$$

(Of course these slopes are also defined at $2r=1$ and $n=2$, but we leave those special cases aside because they lead to simple and double roots in Weierstrass's discussion, already treated above.) The slope of the $\tau(K)$ plot at $K=\sqrt{rJ_m/(r+n)}$ is always greater than μ_0. It follows that between $K=0$ and $\sqrt{rJ_m/(r+n)}$, the solid is strain softening in shear, and that between $K=\sqrt{rJ_m/(r+n)}$ and $\sqrt{J_m}$, the solid is strain hardening in shear.

Third, we make sure that the shear response is a monotonically increasing function of K, by taking γ small enough so that the equation $\tau'(K)=0$ has no root in the $[0,\sqrt{J_m}]$ interval. Note that the actual value of γ does not affect the nondimensional equation of motion (26).

Figure 2 displays some examples of shear stress responses satisfying the requirements just evoked.

Having checked that the shear response is sound, we may now look for solutions to the nondimensional equation (26) describing solitary wave solutions to the original equation (20); these are homoclinic or heteroclinic solutions to (26).

Owing to (28), the barrier $\bar{\omega}=1$ is necessarily an asymptotic point. Taking $r\geqslant 1$ also creates an asymptotic point barrier $\bar{\omega}=0$, leading to a solitary kink with tails of infinite extend. However, taking $1/2<r<1$ gives a barrier reachable in a finite time. The result is a semicompact wave, coming from value 1 at $-\infty$ and decreasing to zero, which it reaches in a finite time with zero speed and zero acceleration. It may then remain at this zero value. For Fig. 3 we took the case $r=3/4$, $n=3$, and chose $\omega(0)=0$ to fix the value of the integration constant.

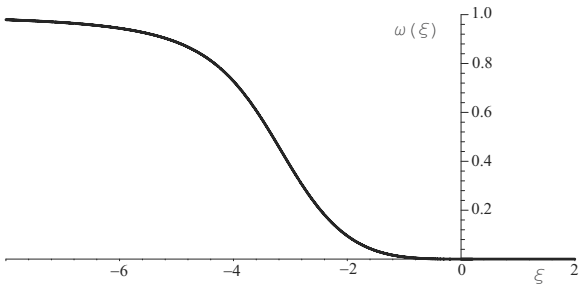

FIG. 3. Semicompact solitary kink wave in a dispersive nonlinear solid with shear stress response given by (27) at $r=3/4$, $n=3$.

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