ON THE ABAQUS FEA MODEL OF FINITE VISCOELASTICITY

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ABSTRACT

Predictions of the QLV (Quasi-Linear Viscoelastic) constitutive law are compared with those of the ABAQUS viscoelastic model for two simple motions in order to highlight, in particular, their very different dissipation rates and certain shortcomings of the ABAQUS model.

INTRODUCTION

As the demand for fuel efficient cars intensifies, tyre manufacturers are trying harder to estimate accurately the energy dissipation of their products. However, the mechanics of tires is a most complex topic due to the variety of mechanical factors involved, such as structural inhomogeneities, complicated geometries and nonlinear material behavior, including hyperelasticity, viscoelasticity and hypoelasticity, for example. Consequently, advanced finite element codes are often called upon to simulate the behavior of tires in real-world applications, and, in particular, to evaluate their energy efficiency. These commercial codes are often used as "black boxes", and the validity of the results is rarely questioned, even though they might provide a decisive argument in favor of, or against, the viability of a given tire model.

In this note we examine the current implementation of nonlinear viscoelastic effects in the ABAQUS Finite Element Analysis (FEA) package because we have noticed certain discrepancies in the model used.** In particular, the ABAQUS model leads to results that are not consistent with the thermomechanically-based Quasi-Linear Viscoelastic (QLV) model. In Section "Finite Viscoelasticity Models" we therefore present both models and highlight their main differences. We then investigate two prototype experiments, for which purpose we use the incompressible Yeoh material for the elastic part of the material response in each case. In Section "Simple Tension" we discuss the uniaxial tension test, while in Section "Simple Shear" the simple shear test is examined. The ABAQUS model predicts somewhat different stress components for each of the considered motions and, more significantly, considerably larger dissipation rates than the QLV model.

FINITE VISCOELASTICITY MODELS

First we introduce some notations. We denote by $\mathbf{F}$ the deformation gradient, which is defined by $\mathbf{F}= \partial\mathbf{x}/\partial\mathbf{X}$, where $\mathbf{x}$ is the position vector in the current configuration of a particle located at position $\mathbf{X}$ in the reference configuration. Let J denote its determinant ($J = \det \mathbf{F}$) and $\mathrm{B} \equiv \mathbf{F}\mathbf{F}^T$, $\mathbf{C} \equiv \mathbf{F}^T\mathbf{F}$ the associated (left and right) Cauchy-Green deformation tensors, where a

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** A new version of the Abaqus Finite Element Analysis package has recently been released (June 2009) and the finite visoelasticity model has been replaced.

superscript ^{T} signifies the transpose. We also introduce the velocity gradient $\mathbf{D} \equiv {}^1/_2\dot{\mathbf{F}}\mathbf{F}^{-1}+{}^1/_2(\dot{\mathbf{F}}\mathbf{F}^{-1})^T]$, where the superimposed dot represents the material time derivative. For an *incompressible* material, every motion is isochoric, so that

$$J \equiv 1,\ \mathrm{tr}\mathbf{D} \equiv 0 \tag{1}$$

We recall that the internal rate of working of the stress per unit current volume is $\sigma(t) \bullet \mathbf{D}(t) \equiv \mathrm{tr}[\sigma(t)\mathbf{D}(t)]$ where $\sigma(t)$ is the Cauchy stress tensor at time t. Assuming for $\mathbf{D}(t)$ a periodic time law, the time-averaged work E_d done per unit current volume over a period T starting at time T_0, is given by

$$E_d = \frac{1}{T}\int_{T_0}^{T_0+T} \sigma(t)\cdot\mathbf{D}(t)\mathrm{d}t \tag{2}$$

Since the elastic part of the stress does not contribute to this expression (2) represents the energy dissipated in a cycle (which depends on T_0 in general).

THE ABAQUS MODEL

Next, we present the ABAQUS model. Section 4.8.2 of the ABAQUS Theory Manual (Hibbit *et al.* 2007) gives the constitutive relation for modeling nonlinear viscoelastic effects in the form

$$\sigma(t) = \sigma_e(t) + \mathrm{SYM}\left\{\mathbf{F}(t)\left[\int_0^t \frac{J(s)}{J(t)}\dot{G}(t-s)\mathbf{F}^{-1}(\mathrm{s})\sigma_{\mathrm{e}}(\mathrm{s})\mathbf{F}(\mathrm{s})\mathrm{ds}\right]\mathbf{F}^{-1}(t)\right\} \tag{3}$$

where σ_e is the instantaneous Cauchy stress response (elastic response at very short times), G is the so-called memory kernel, which characterizes the stress relaxation and satisfies $G(0) = 1$, and $\dot{G}$ denotes the derivative of G with respect to its argument t-s. Also, "SYM" represents the symmetric part of the bracketed term; *e.g.*, $\mathbf{D} = \mathrm{SYM}\{\dot{\mathbf{F}}\mathbf{F}^{-1}\}$. The constitutive relation (3) is valid for compressible as well as incompressible solids since in the latter case the hydrostatic term $-\hat{p}\mathbf{I}$ in σ_e (where $\hat{p}$ is a Lagrange multiplier) is a workless constraint stress in both the instantaneous response and in the history term, as expected. Indeed, for an incompressible solid $J = 1$ for all times and σ_e has the general form

$$\sigma_e = -\hat{p}\mathbf{I} + \Psi_1\mathbf{B} + \Psi_2\mathbf{B}^2 \tag{4}$$

where Ψ_1, Ψ_2 are scalar functions of time and of the first and second principal invariants, I_1, I_2 of $\mathbf{C}$ (or equivalently $\mathbf{B}$), which, for an incompressible material, are defined by I_1= tr $\mathbf{B}$, I_2= tr ($\mathbf{B}^{-1}$). Then (3) reduces to

$$\begin{aligned}\sigma(t) &= -p(t)\mathbf{I} + \Psi_1(t)\mathbf{B}(t) + \Psi_2(t)\mathbf{B}(t)^2\\ &+\sum_{i=1}^{2}\mathrm{SYM}\left\{\mathbf{F}(t)\left[\int_0^t \dot{G}(t-s)\Psi_i(s)\mathbf{C}(s)^i\mathrm{d}s\right]\mathbf{F}^{-1}(t)\right\}\end{aligned} \tag{5}$$

where $p(t) = \hat{p}(t)+\int_0^t \dot{G}(t\text{-}s)\hat{p}(s)\mathrm{d}s$ is arbitrary and remains to be determined from initial/boundary conditions.

THE QLV MODEL

The ABAQUS model is reminiscent of, and similar to, a well-established model of finite viscoelasticity, namely the Pipkin – Rogers model (Pipkin and Rogers 1968), which reads in the incompressible case as

$$\sigma(t) = -p(t)\mathbf{I} + \mathbf{F}(t)\left\{\mathbf{R}\left[\mathbf{C}(t),0\right] + \int_0^t \frac{\partial}{\partial(t-s)}\left(\mathbf{R}\left[\mathbf{C}(s),t-s\right]\right)\mathrm{d}s\right\}\mathbf{F}(t)^T \qquad (6)$$

(see, for example, Wineman 1972, Johnson *et al.* 1996). Here $p(t)$ is a Lagrange multiplier resulting from the internal constraint of incompressibility and $\mathbf{R}$ is a strain-dependent tensorial relaxation function having the form

$$\mathbf{R}\left[\mathbf{C}(s),t-s\right] = \phi_0(s,t-s)\mathbf{I} + \phi_1(s,t-s)\mathbf{C}(s) + \phi_2(s,t-s)\mathbf{C}(s)^2 \qquad (7)$$

where ϕ_0, ϕ_1, ϕ_2 are scalar functions of time t-s and of I_1 and I_2 at time s.

With an appropriate choice of ϕ_0, ϕ_1, ϕ_2, the Pipkin-Rogers model reduces to the so-called Quasi-Linear Viscoelastic (QLV) model, which has proved to be a successful phenomenological model for describing the nonlinear viscoelastic behavior of solids (see references in Johnson *et al.* 1996, Rajagopal and Wineman 2008, for example). Equation (6) is derived rigorously by successive approximations from the basic physical requirements governing the behavior of solids with memory, such as the principle of determinism and local action and the principle of material objectivity (see the review by Drapaca et al. 2007 and references therein).

On inspection of (5) and (6) it can be seen that there are two main differences between the models. First, the integral term in equation (3) is generally non-symmetric, in contrast to the integral term in equation (6). This is taken care of in an *ad hoc* manner by using the "SYM" operator. Also, the history (time integral) term in the ABAQUS model terminates with $\mathbf{F}(t)^{-1}$ in contrast to the history term in the the QLV model, which terminates with $\mathbf{F}(t)^{\mathrm{T}}$. The latter fits more naturally with the usual expression for the traction $\sigma\mathbf{n}\mathrm{d}a$ via Nanson's formula $\mathbf{F}^{\mathrm{T}}\,\mathbf{n}\mathrm{d}a = J\mathbf{N}\mathrm{d}A$ connecting reference and deformed area elements ($J = 1$ here). In fact, the 'push forward' to the configuration at time t from that at time s of the (symmetric) 'Cauchy' stress $\sigma_e(s)$ should involve $\mathbf{F}(t)\mathbf{F}(s)^{-1}\sigma_e(s)\mathbf{F}(s)^{-T}\mathbf{F}(t)^T$ rather than the $\mathbf{F}(t)\mathbf{F}(s)^{-1}\sigma_e(s)\mathbf{F}(s)\mathbf{F}(t)^{-1}$ that appears in (3). This change would remove the need to apply the SYM operation. However, for an incompressible material use of (4) then leads to a term in $\hat{p}$ that doesn't give a workless constraint stress. This can be corrected by, for example, dropping this term from (4) in the integral, in which case (5) would be replaced by

$$\begin{aligned}\sigma(t) &= -p(t)\mathbf{I} + \Psi_1(t)\mathbf{B}(t) + \Psi_2(t)\mathbf{B}(t)^2 \\ &+\mathbf{F}(t)\left\{\int_0^t \dot{\mathrm{G}}(t-s)\left[\Psi_1(s)\mathbf{I} + \Psi_2\mathrm{C}(s)\right]\mathrm{d}s\right\}\mathbf{F}^T(t)\end{aligned} \qquad (8)$$

with $p(t)$ the arbitrary pressure. This is then a special case within the model (6).

To emphasize the differences between the models, we henceforth focus on an incompressible viscoelastic solid for which the instantaneous response is modeled by a two-term Yeoh stress-strain relationship (a standard implementation in ABAQUS)

$$\sigma_e = -p\mathbf{I} + \mu_0\left(1 - 3\alpha + \alpha \mathrm{I}_1\right)\mathbf{B} \tag{9}$$

where μ_0 and α are positive constants (μ_0 is the shear modulus in the reference configuration) and $I_1(t) = \mathrm{tr}[\mathbf{B}(t)]$. Also, for simplicity, the time relaxation of the solid is assumed to be governed by a one-term Prony series expansion given by

$$G(t) = \frac{\mu_\infty}{\mu_0} + \left(1 - \frac{\mu_\infty}{\mu_0}\right)e^{-t/\tau} \tag{10}$$

where μ_∞ is the ultimate value to which the shear modulus settles after an infinite time and τ is a characteristic time constant.

Then, for the ABAQUS model we have $\Psi_1(t) = \mu_0[1\text{-}3\alpha + \alpha I_1(t)]$, $\Psi_2(t) = 0$, and hence (5) reduces to

$$\begin{aligned}\sigma(t) &= -p(t)\mathbf{I} + \mu_0\left[1 - 3\alpha + \alpha I_1(t)\right]\mathbf{B}(t)\\ &+ \frac{\left(\mu_\infty - \mu_0\right)}{\tau} SYM\left\{\mathbf{F}(t)\left[\int_0^t e^{-(t-s)/\tau}\left[1 - 3\alpha + \alpha I_1(s)\right]\mathbf{C}(s)\mathrm{d}s\right]\mathbf{F}^{-1}(t)\right\}\end{aligned} \tag{11}$$

By contrast, for the QLV model we have

$$\phi_0(s, t-s) = \mu_0\left[1 - 3\alpha + \alpha I_1(s)\right]G(t-s),\ \phi_1(s, t-s) = 0,\ \phi_2(s, t-s) = 0$$

yielding

$$\begin{aligned}\sigma(t) &= -p(t)\mathbf{I} + \mu_0\left[1 - 3\alpha + \alpha I_1(t)\right]\mathbf{B}(t)\\ &+ \frac{\left(\mu_\infty - \mu_0\right)}{\tau}\left[\int_0^t e^{-(t-s)/\tau}\left[1 - 3\alpha + \alpha I_1(s)\right]ds\right]\mathbf{B}(t)\end{aligned} \tag{12}$$

In this specialization the modification (8) of the ABAQUS model gives exactly the same expression for $\sigma(t)$ as (12). We now examine two simple examples of motions.

SIMPLE TENSION

In a simple tension test of an incompressible isotropic solid the (homogeneous) motion is described by

$$x_1 = \lambda(t)X_1,\ x_2 = \lambda(t)^{-1/2}X_2,\ x_3 = \lambda(t)^{-1/2}X_3 \tag{13}$$

where $\lambda(t)$ (≥ 1) is the stretch ratio in the direction of the uniaxial tension σ_{11}(≥ 0). The resulting deformation gradient has the diagonal form

$$\mathbf{F}(t) = \mathrm{Diag}\left[\lambda(t), \lambda(t)^{-1/2}, \lambda(t)^{-1/2}\right] \tag{14}$$

Vanishing of the lateral stresses $\sigma_{22} = \sigma_{33} = 0$ determines the Lagrange multiplier *p(t)*, and elimination of *p(t)* yields

$$\begin{aligned}\sigma_{11}(t) &= \mu_0\left[\lambda(t) - \lambda^{-2}(t)\right]\left[2\alpha + (1-3\alpha)\lambda(t) + \alpha\lambda^3(t)\right] \\ &+ \frac{(\mu_\infty - \mu_0)}{\tau}\int_0^t e^{-(t-s)/\tau}\left[\lambda(s) - \lambda^{-2}(s)\right]\left[2\alpha + (1-3\alpha)\lambda(s) + \alpha\lambda^3(s)\right]\mathrm{d}s\end{aligned} \tag{15}$$

for the ABAQUS model, and

$$\begin{aligned}\sigma_{11}(t) &= \mu_0\left[\lambda(t) - \lambda^{-2}(t)\right]\left[2\alpha + (1-3\alpha)\lambda(t) + \alpha\lambda^3(t)\right] \\ &+ \frac{(\mu_\infty - \mu_0)}{\tau}\left[\lambda^2(t) - \lambda^{-1}(t)\right]\int_0^t e^{-(t-s)/\tau}\lambda^{-1}(s)\left[2\alpha + (1-3\alpha)\lambda(s) + \alpha\lambda^3(s)\right]\mathrm{d}s\end{aligned} \tag{16}$$

for the QLV model. The difference between these models is now clear, and it reflects significantly on the rate of working, as we confirm numerically below.

From an experimental point of view, it is common practice to employ a dynamic displacement superimposed on a large static deformation. Here we consider that the Yeoh solid is deformed in tension to a stretch of 1.3 from time t = 0 to time t = 1, and then made to oscillate with a superimposed amplitude such that the stretch $\lambda(t)$ ranges from 1.1 to 1.5. Thus,

$$\lambda(t) = \begin{cases} 1 + 0.3t & 0 \le t \le 1 \\ 1.3 + 0.2\sin\omega(t-1) & t \ge 1 \end{cases} \tag{17}$$

For numerical purposes the stresses are non-dimensionalized by dividing by μ_0. For the remaining parameters we set

$$\alpha = 1.0, \quad \mu_\infty/\mu_0 = 0.5, \quad \tau = 0.01s, \quad \omega = 16\pi s^{-1} \tag{18}$$

except that we retain ω as a free parameter in investigating the frequency dependence of the energy dissipation rate. For these values Figure 1 shows how the predictions of the ABAQUS model for the dimensionless stress $\sigma_{11}(t)/\mu_0$ differ from those of the QLV model. This difference is not huge, but, in contrast, the predictions of the energy dissipation are very different, as we now illustrate.

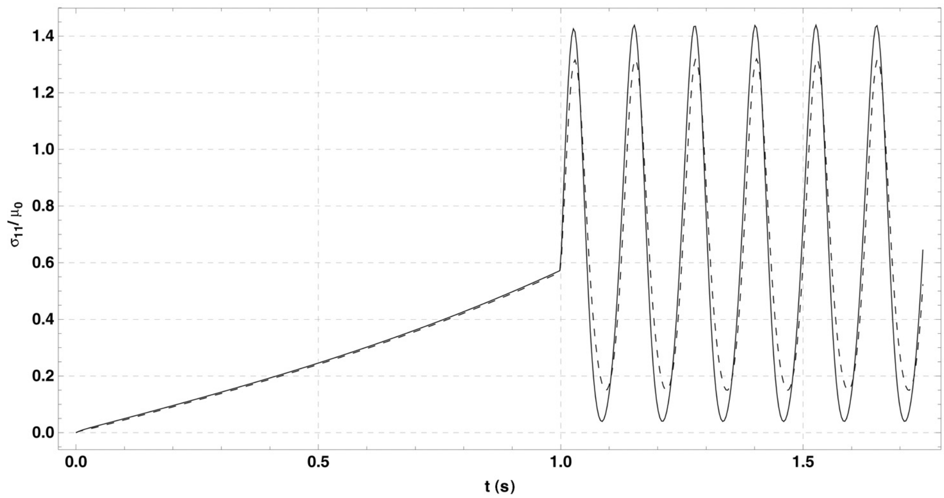

FIG. 1. — Dependence of the dimensionless axial tension on time for the ABAQUS FEA model (solid curve), the ABAQUS output (dotted curve) and the QLV model (dashed curve).

The numerical results have been obtained from ABAQUS 6.7-1 using a single CPS4 element (4 nodes, bilinear, plane stress) with an implicit solution scheme. As a check the same test was done with a 3D brick element C3D8H (8 nodes, linear, hybrid) and the same results were obtained. Note that in both cases, we used a single element and since the deformation is uniform, the integration is always exact and it doesn't depend on the element order.

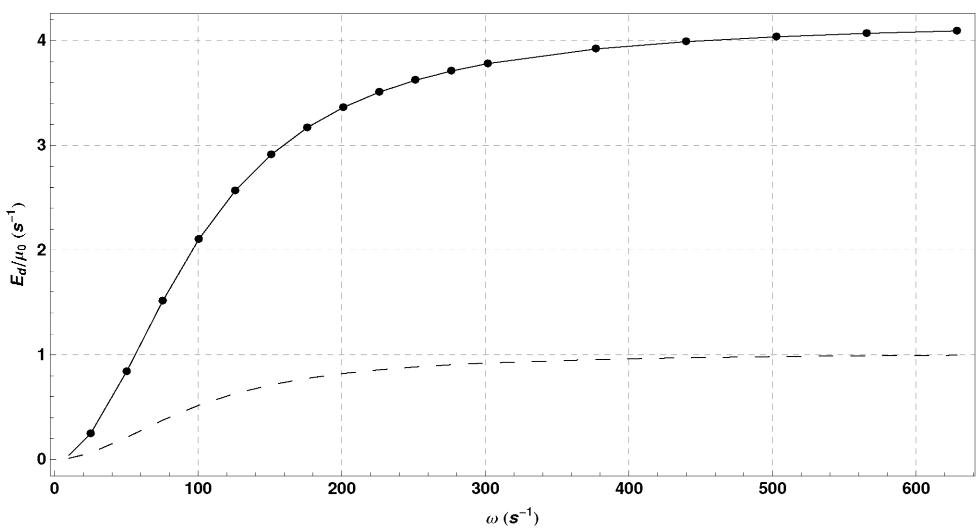

FIG. 2. — Dependence of the rate of working per unit volume in dimensionless form on frequency over a single period in the steady state for the ABAQUS model (solid curve), for the ABAQUS output (dotted curve) and for the QLV model (dashed curve) in the case of uniaxial tension.

The results settle very rapidly to the steady state so that the expression (2) becomes essentially independent of T_0. Figure 2, in which E_d/μ_0 is plotted against frequency in the steady state, shows clearly that the ABAQUS model overestimates the steady state energy dissipation substantially (by a factor of 4 to 5) compared with the QLV model.

SIMPLE SHEAR

We now consider a simple shear motion of the form

$$x_1 = X_1 + \gamma(t)X_2,\ \ x_2 = X_2,\ \ x_3 = X_3 \tag{19}$$

where $\gamma(t)$ is the amount of shear. The (non-symmetric) deformation gradient has components

$$F(t) = \begin{bmatrix} 1 & \gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

Simple calculations reveal that a sheared solid described by the ABAQUS model or by the QLV model is in a state of *plane stress* ($\sigma_{ij} \neq 0$ for $i, j \in \{1,2\}$; $\sigma_{3j} = 0$ for $j = 1,2,3$) when the Lagrange multiplier is taken as

$$p(t) = \square_\infty - (\square_\infty - \square_0)e^{-t/\tau}\alpha\square_0\gamma^2(t) - \alpha\frac{(\square_\infty - \square_0)}{\tau}\int_0^1 e^{-(t-s)/\tau}\gamma^2(s)\mathrm{d}s \tag{21}$$

Note that the combination of plane strain and plane stress (in the plane) is permissible for an incompressible material provided $p(t)$ (and hence $\sigma_{11}(t)$ and $\sigma_{22}(t)$) is (are) adjusted accordingly. Then, for the ABAQUS model we obtain

$$\begin{aligned}
\sigma_{11}(t) &= \square_0\gamma^2(t)\left[1 + \alpha\gamma^2(t)\right] + \frac{(\square_\infty - \square_0)}{\tau}\gamma(t)\int_o^t e^{-(t-s)/\tau}\gamma(s)\left[1 + \alpha\gamma^2(s)\right]\mathrm{d}s \\
\sigma_{12}(t) &= \square_0\gamma(t)\left[1 + \alpha\gamma^2(t)\right] \\
&+ \frac{(\square_\infty - \square_0)}{2\tau}\int_0^t e^{-(t-s)/\tau}\gamma(s)\left[2 + \gamma(s)\gamma(t) - \gamma^2(t)\right]\left[1 + \alpha\gamma^2(s)\right]\mathrm{d}s \\
\sigma_{22}(t) &= \frac{(\square_\infty - \square_0)}{\tau}\int_0^t e^{-(t-s)/\tau}\gamma(s)\left[\gamma(s) - \gamma(t)\right]\left[1 + \alpha\gamma^2(s)\right]\mathrm{d}s
\end{aligned} \tag{22}$$

and for the QLV model

$$\begin{aligned}
\sigma_{11}(t) &= \square_0\gamma^2(t)\left[1 + \alpha\gamma^2(t)\right] + \frac{(\square_\infty - \square_0)}{\tau}\gamma^2(t)\int_o^t e^{-(t-s)/\tau}\left[1 + \alpha\gamma^2(s)\right]\mathrm{d}s \\
\sigma_{12}(t) &= \square_0\gamma(t)\left[1 + \alpha\gamma^2(t)\right] + \frac{(\square_\infty - \square_0)}{\tau}\gamma(t)\int_o^t e^{-(t-s)/\tau}\left[1 + \alpha\gamma^2(s)\right]\mathrm{d}s \\
\sigma_{22}(t) &= 0
\end{aligned} \tag{23}$$

Note that Rivlin's universal relation σ_{11}-$\sigma_{22} = \gamma\sigma_{12}$ from isotropic elasticity holds also for the present Yeoh-based viscoelastic model. The expressions in (23) are expected intuitively for the Yeoh model because for its instantaneous response at very short times the Cauchy stress has components $\sigma_{e11} = \mu_0\gamma^2(1+\alpha\gamma^2)$, $\sigma_{e12} = \mu_0\gamma(1+\alpha\gamma^2)$, $\sigma_{e22} = 0$. By contrast, the ABAQUS model gives rise to a component generated purely by the viscoelastic effects, in which case the relaxation process creates such a component *ex nihilo*! However, since simple shear is a displacement controlled motion the stress components adjust automatically to accommodate the geometry and they are therefore very much dependent on the form of the constitutive law. The difference in the shear stress, however, has serious consequences for the rate of working, and hence the dissipation, because here

$$\mathbf{D}(t) = \begin{bmatrix} 0 & \dot{\gamma}(t)/2 & 0 \\ \dot{\gamma}(t)/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (24)$$

and hence $\sigma \bullet \mathbf{D} = \sigma_{12}(t)\dot{\gamma}(t)$, which is obviously not the same for both models.

We confirm these findings by testing the ABAQUS software against the formulas (22)–(24). We take the amount of shear to vary as

$$\gamma(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1 + 0.2\sin[\omega(t-1)] & t \ge 1 \end{cases} \qquad (25)$$

with the other parameters given by (18). Figures 3 and 4 display the variations of the σ_{12} and σ_{22} components computed from (22) and (23) in dimensionless form. As in the case of simple tension there is not a great difference in the active stress (in this case σ_{12}) between the two models, but the reactive stress σ_{22} is very different. Finally, as also for simple tension, we find that the ABAQUS model overestimates the rate of working with respect to the QLV model (by a factor of 2 to 3), as shown in Figure 5, again in dimensionless form with E_d/μ_0 plotted against frequency ω in the steady state. Using the ABAQUS software, we recovered the thick curves (in Figures 3 and 4), which confirms that equation (3) is actually implemented in the ABAQUS code.

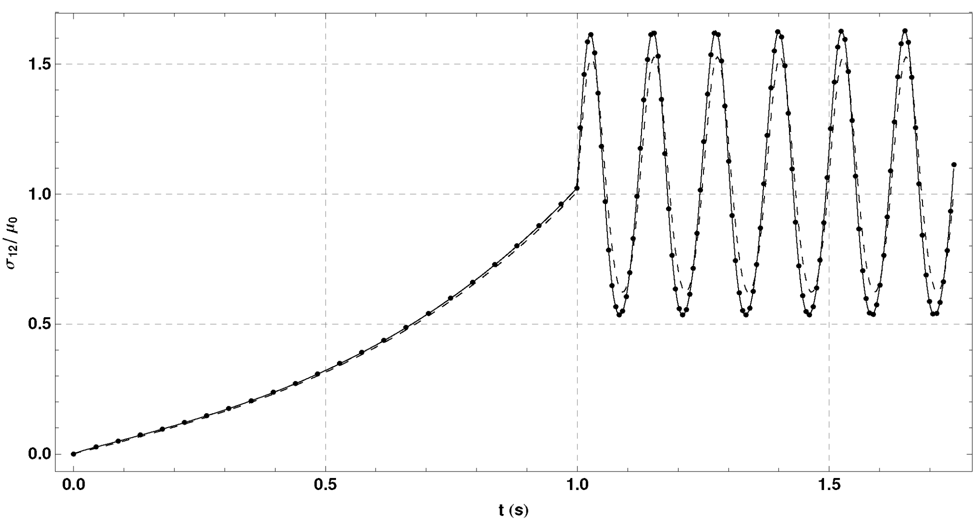

FIG. 3. — Dependence of the dimensionless shear stress on time in simple shear for the ABAQUS analytical model (solid curve), the ABAQUS output (dotted curve) and the QLV model (dashed curve).

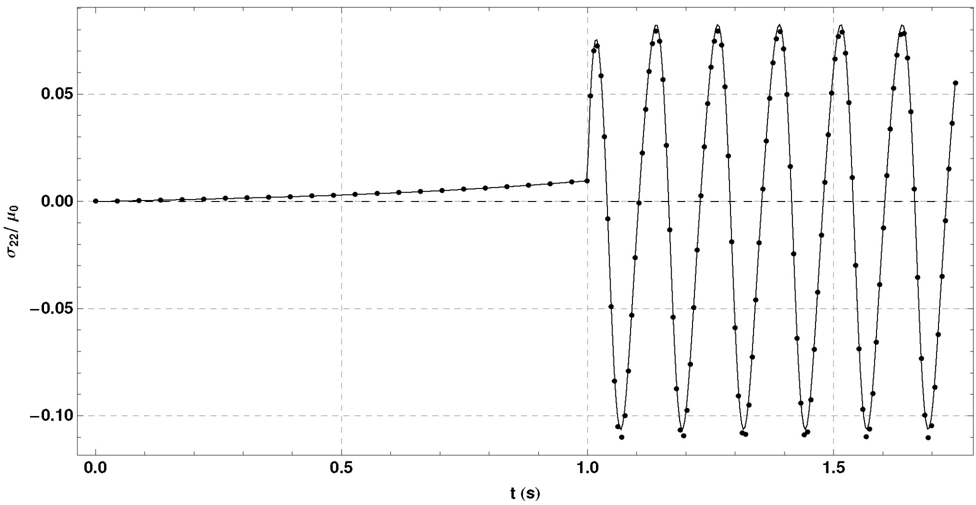

FIG. 4. — Dependence of the dimensionless normal stress in 2-direction on time in simple shear for the ABAQUS FEA model (solid curve), the ABAQUS output (dotted curve) and the QLV model (dashed curve). The latter is zero for all times.

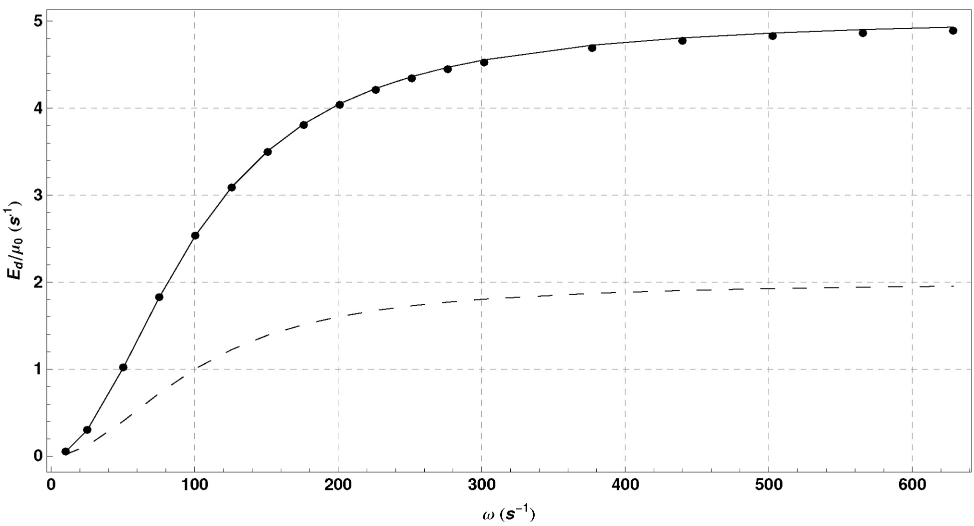

FIG. 5. — Dependence of the rate of working per unit volume in dimensionless form on frequency over a single period in the steady state for the ABAQUS model (solid curve), for the ABAQUS output (dotted curve) and for the QLV model (dashed curve) in the case of simple shear.

ACKNOWLEDGEMENTS

We thank the Università di Roma "La Sapienza" (Italy) and the Royal Society (UK) for support in part via an International Joint Project Grant. The work of M.D. is supported by a Senior Marie Curie Fellowship from the European Commission (Seventh Framework Programme).

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[*Received June 2008, revised May 2009*]