Onset of Nonlinearity in the Elastic Bending of Blocks

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The classical flexure problem of nonlinear incompressible elasticity is revisited assuming that the bending angle suffered by the block is specified instead of the usual applied moment. The general moment-bending angle relationship is then obtained and is shown to be dependent on only one nondimensional parameter: the product of the aspect ratio of the block and the bending angle. A Maclaurin series expansion in this parameter is then found. The first-order term is proportional to μ, the shear modulus of linear elasticity; the second-order term is identically zero because the moment is an odd function of the angle; and the third-order term is proportional to $\mu(4\beta-1)$, where β is the nonlinear shear coefficient, involving third-order and fourth-order elasticity constants. It follows that bending experiments provide an alternative way of estimating this coefficient and the results of one such experiment are presented. In passing, the coefficients of Rivlin's expansion in exact nonlinear elasticity are connected to those of Landau in weakly (fourth-order) nonlinear elasticity. [DOI: 10.1115/1.4001282]

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1 Introduction

The problem of flexure is one of the now classical problems of the theory of nonlinear, incompressible elasticity. First formulated and solved by Rivlin [1], it has since been studied extensively in the literature, see for example, Green and Zerna [2] and Ogden [3]. There is continuing theoretical interest in this problem, as can be seen for example, in the recent study of Kanner and Horgan [4]. The physical problem considered is easily visualized: a rectangular specimen is bent by equal and opposite terminal couples applied on the end faces of a rectangular block while its other faces remain free of traction. Rivlin [1] showed that if a circular, annular sector is assumed for the deformed configuration, then an elegant solution to the corresponding boundary value problem can be found. We recall this derivation in Secs. 2 and 3.

Typically, the usual formulation of this boundary value problem implicitly assumes that the terminal moments are specified. It is shown here that specifying instead the *bending angle* through which the block is bent results in a simpler mathematical formulation and solution of the problem. We show that this solution depends on only one nondimensional parameter: the product of the aspect ratio of the block and the bending angle, which we denote by ϵ. If ϵ is assumed small, then the solution at low orders has a particularly simple form. Because ϵ is the product of the aspect ratio of the block and the bending angle, the lower-order solutions are applicable in at least two distinct physical regimes: the first corresponds to the bending of bars through an infinitesimal bending angle and the second corresponds to the nonlinear bending of thin sheets. Although these two subcases are the most important, there are other possibilities: moderately thick blocks could also be considered turned through moderate angles, provided the product ϵ is small.

To analyze the problem, *plane strain* conditions can be assumed. Hence, the technical interpretation of our results is that they describe the bending of planar sections of infinitely wide blocks. We assume that the plane strain results obtained are also applicable to strips with a finite, out of the plane dimension larger than the block thickness, i.e., that the edge/anticlastic effects are negligible. This assumption is motivated by observations and measurements for the bending of rubber blocks, see Gent and Cho [5]. Of course, this assumption is a limitation of Rivlin's solution because secondary fields are bound to be observed outside of a central area in the bent block, see Fig. 1. Nonetheless, it must be kept in mind that Rivlin's solution is one of the very few *universal* solutions, valid in principle for every incompressible isotropic material, whatever the actual dimensions of the block and the amount of bending. For more refined deformations, albeit limited to certain types of geometries, the reader is referred to the advances obtained by Shield [6]. The other limitation of this solution is that it might bifurcate, see Refs. [7–10] for an in-depth treatment of this possibility.

Given that here, the deformation depends only on the bending angle, the corresponding stress distribution can be easily determined. The most important functional of this stress distribution is the moment that needs to be applied at the ends of the block to effect the deformation. In Secs. 4 and 5, we show that the general relation between applied moment and the parameter ϵ, the main relation of interest in this problem, can be expressed in a succinct and elegant form. It is shown that this relation is an odd function in ϵ, which agrees with an intuitive expectation that the moment should be an odd function of the angle since the moment required to bend a block by an angle α say, is the opposite of the moment required to bend it by an angle $-\alpha$.

On expanding the moment in a Maclaurin series in ϵ (Sec. 6), we find that the first-order coefficient is proportional to μ, the shear modulus of linear elasticity. The second-order coefficient is identically zero for all elastic materials because the moment is an odd function in ϵ. This suggests that the linearized moment-angle relation is likely to be valid for ϵ values beyond the infinitesimal range, and this is verified numerically (see Fig. 3) and experimentally (Table 1and Fig. 4). in Sec. 8. Prior to this, we show in Sec. 7 that the third-order term in the expansion is proportional to $\mu(4\beta-1)$, where β is the nonlinear coefficient of plane shear waves [11–14]. Explicitly, $2\beta=(\mu+\mathcal{A}/2+\mathcal{D})/\mu$, where $\mathcal{A}$ and $\mathcal{D}$ are Landau third- and fourth-order elastic constants [15]. This coefficient has been measured for agar-based gels, based on the measurement of shear wave speeds in transient elastography [14], or on the measurement of homogeneous plane strain deformations [16]. Clearly, the information collected from the bending of a

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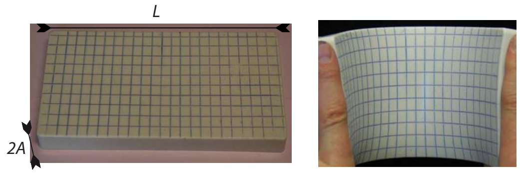

Fig. 1 Bending of a block of silicone rubber with length L =24 cm and thickness $2A$=2 cm. The dimension of each square drawn on the top surface is 1×1 cm^2. The picture on the right shows that even for such a "homemade" bending experiment, there exists a region about 3 cm wide around the median where plane strain is respected.

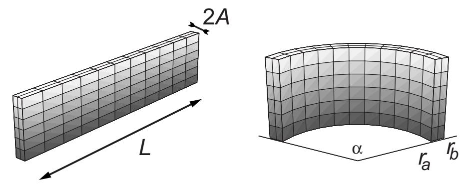

Fig. 2 Sketch of Rivlin's deformation for the bending of a block made of an incompressible isotropic solid with length L and thickness $2A$ into a circular annular sector with inner and outer radii r_a and r_b, respectively. The angle α is the bending angle. The deformation is plane strain but makes no assumption about the dimensions of the block or the amount of bending.

block, such as that provided by a bending stiffness tester [17], yields a simple and useful alternative to these protocols.

In Sec. 8, our main results are then compared with experimental data obtained by performing a bending experiment on a polyurethane elastomer. We find that $\beta \simeq 1.0$.

2 Large Plane Strain Bending

The fundamental assumption introduced by Rivlin [1] to model the nonlinear flexure of an incompressible block is that a block of length L and thickness $2A$ is deformed under applied terminal moments into a circular, annular sector. For definiteness, assume that the faces $X=\pm A$ are deformed into the inner and outer radii, denoted by r_a and r_b, respectively, of the annular sector and the faces $Y=\pm L/2$ are deformed into the faces $\theta=\pm\alpha$, where α, the bending angle, is a *specified* constant. Plane strain conditions are assumed throughout. The bending angle α is restricted to lie in the range

$$0 \le \alpha \le \pi \tag{2.1}$$

which only allows a block to be bent into at most a circular annulus, see Fig. 2.

Adopting the semi-inverse approach of Rivlin [1], assume that

$$r = r(X), \quad \theta = \theta(Y), \quad z = Z \tag{2.2}$$

where (X,Y,Z) and (r,θ,z) denote the Cartesian and cylindrical polar coordinates of a typical particle before and after deformation, respectively. Incompressibility then yields

$$r^2 = 2BX + D, \quad \theta = Y/B + C, \quad z = Z \tag{2.3}$$

where B, C, and D are constants.

As noted by Rivlin [1], symmetric boundary value problems can be considered without loss of generality and therefore $C\equiv 0$. The key element in our solution of the bending problem is that the bending angle α is specified and not the applied moment, as is usually assumed in most treatments, even if this assumption is implicit. It therefore follows easily that B can be *determined* as

$$B = L/\alpha \tag{2.4}$$

This yields the nonhomogeneous deformation field

$$r = \sqrt{2(L/\alpha)X + D}, \quad \theta = (\alpha/L)Y, \quad z = Z \tag{2.5}$$

where D remains to be determined. Therefore, the inner and outer radii of the deformed curved surfaces are determined by

$$r_{a,b} = \sqrt{D \mp 2(L/\alpha)A} \tag{2.6}$$

Adding and subtracting these equations then yields

$$D = (r_a^2 + r_b^2)/2 \tag{2.7a}$$

$$r_b^2 - r_a^2 = 4AL/\alpha \tag{2.7b}$$

Hereafter we consider the boundary value problem where equal and opposite moments are applied to the ends of the block at $Y = \pm L/2$. The other classical boundary value problem of flexure, where one end is held fixed and a moment applied to the free end, is a subregion of the problem considered here with the fixed end described by $\theta=0$.

3 The Stress-Free Boundary Conditions

The corresponding deformation gradient tensor $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3) = \mathrm{diag}(\lambda,\lambda^{-1},1) \tag{3.1a}$$

where

$$\lambda = L/(\alpha r) \tag{3.1b}$$

denoting the principal stretches by λ_1, λ_2, and λ_3. It is now clear that Rivlin's solution (2.5) is a plane strain deformation because $\lambda_3=1$ at all times.

For homogeneous, incompressible, elastic materials, the corresponding principal Cauchy stresses are given by

$$T_{rr} = -p + \lambda_1 W_{,1}, \quad T_{\theta\theta} = -p + \lambda_2 W_{,2} \tag{3.2}$$

where p is an arbitrary scalar field, $W=W(\lambda_1,\lambda_2,\lambda_3)$ is the strain-energy function and the comma subscript denotes partial differentiation with respect to the appropriate principal stretch. The equations of equilibrium determine p as

$$p = \int (\lambda_1 W_{,1} - \lambda_2 W_{,2}) r^{-1} \mathrm{d}r + \lambda_1 W_{,1} + K \tag{3.3}$$

where K is an arbitrary constant. It therefore follows immediately that

$$T_{rr} = \int (\lambda_1 W_{,1} - \lambda_2 W_{,2}) r^{-1} \mathrm{d}r + K, \quad T_{\theta\theta} = T_{rr} + \lambda_2 W_{,2} - \lambda_1 W_{,1} \tag{3.4}$$

Now define the function $\widetilde{W}(\lambda)$ as

$$\widetilde{W}(\lambda) = W(\lambda, \lambda^{-1}, 1) \tag{3.5}$$

which is assumed to be a convex function. Then

Table 1 Experimental results collected for the bending of a polyurethane strip with aspect ratio $A/L \simeq 0.35$. The first line gives the bending angle α in degrees and the second line gives a measure of the moment M up to a multiplicative factor.

α	3	4	5	7	10	15	20	25	30	35	40	45	50	55	60
M/M_m	4	5 1/2	7	10	14	21	28 1/2	35 1/2	43	51	59	67	76	85	93

$$\lambda \widetilde{W}' = \lambda_1 W_{,1} - \lambda_2 W_{,2} \tag{3.6}$$

where the prime denotes differentiation. The stress distribution can then be written simply as functions of λ as

$$T_{rr} = \widetilde{W} + K \tag{3.7a}$$

$$T_{\theta\theta} = \widetilde{W} - \lambda \widetilde{W}' + K \tag{3.7b}$$

The curved surfaces of the bent block are assumed to be free of traction. This assumption then yields

$$K = -\widetilde{W}(\lambda_a) \tag{3.8a}$$

$$\widetilde{W}(\lambda_b) = \widetilde{W}(\lambda_a) \tag{3.8b}$$

where

$$\lambda_a = L/(\alpha r_a), \quad \lambda_b = L/(\alpha r_b) \tag{3.9}$$

No assumptions have been made thus far about material symmetry. Only isotropic materials will be considered here. For these materials, $W(\lambda_1, \lambda_2, 1) = W(\lambda_2, \lambda_1, 1)$ and so Eqs. (3.8a) and (3.8b) yield

$$\begin{aligned} W\left(\frac{L}{\alpha r_a}, \frac{\alpha r_a}{L}, 1\right) &= W\left(\frac{\alpha r_a}{L}, \frac{L}{\alpha r_a}, 1\right) = W\left(\frac{L}{\alpha r_b}, \frac{\alpha r_b}{L}, 1\right) \\ &= W\left(\frac{\alpha r_b}{L}, \frac{L}{\alpha r_b}, 1\right) \end{aligned} \tag{3.10}$$

There are two obvious solutions to these equations: $r_a = r_b$, which is physically unacceptable and [1]

$$\alpha^2 r_a r_b = L^2, \tag{3.11}$$

which is assumed henceforth. Solving for r_b and substitution into Eq. (2.7b) yields a quadratic equation for r_a^2 with the following unique physically acceptable solution:

$$r_a^2 = \frac{L}{\alpha}\left(\sqrt{4A^2 + \frac{L^2}{\alpha^2}} - 2A\right), \tag{3.12}$$

which completely determines the deformed configuration. Unusually, it is *independent* of the form of the strain-energy function (provided that Eq. (3.11), which is sufficient for Eq. (3.10) to be satisfied, is also necessary). Also note that for isotropic materials, it follows immediately from Eq. (3.6) that

$$\widetilde{W}'(1) = 0 \tag{3.13}$$

Substitution of Eqs. (3.11) and (3.12) into Eq. (3.9) yields the following form for the stretches in terms of the nondimensional parameter ϵ:

$$\lambda_a = \sqrt{\sqrt{1+\epsilon^2} + \epsilon}, \quad \lambda_b = \sqrt{\sqrt{1+\epsilon^2} - \epsilon} = \lambda_a^{-1} \tag{3.14}$$

where ϵ is the product of the block aspect ratio by the bending angle

$$\epsilon = \frac{2A}{L}\alpha \tag{3.15}$$

Finally, the qualitative features of the stress distribution will be determined. Equation (3.7a) yields

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} = -\frac{1}{r}\lambda \widetilde{W}'(\lambda), \quad \frac{\mathrm{d}^2 T_{rr}}{\mathrm{d}r^2} = \frac{1}{r^2}(2\lambda \widetilde{W}'(\lambda) + \lambda^2 W''(\lambda)) \tag{3.16}$$

It follows immediately from these, the assumed convexity of $\widetilde{W}$, and Eq. (3.13) that the radial stress has a unique minimum value of $-\widetilde{W}(\lambda_a)$ at $r = L/\alpha$ and since the curved surfaces are assumed stress-free, the radial stress is therefore compressive in the interior of the bent block.

Convexity also yields that the hoop stress is a monotonically increasing function of r, compressive on the inner curved surface and tensile on the outer.

4 Some Approximations of the Deformed Configuration

Since the deformed configuration is independent of the form of the strain-energy function, asymptotic expansions in ϵ are valid for all incompressible elastic materials. Expanding Eq. (3.12) and its counterpart for r_b^2 in a Maclaurin series in ϵ yields

$$r_a = \frac{L}{\alpha}\left[1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)\right], \quad r_b = \frac{L}{\alpha}\left[1 + \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)\right] \tag{4.1}$$

To the leading order, we have

$$r_a = r_b = L/\alpha \tag{4.2}$$

giving a curvature

$$\kappa = \alpha/L \tag{4.3}$$

Thus, infinitesimally thin sheets, strips, and wires are bent into a circular arc whose radius is inversely proportional to the bending angle.

Truncating the series after the linear terms in Eq. (4.1), and thus we are now considering thin but not infinitesimally thin, sheets, strips, and wires yields

$$r_a = L/\alpha - A, \quad r_b = L/\alpha + A \tag{4.4}$$

which are again remarkably simple and which tell us that the thickness of the deformed block is still $2A$.

Retaining the *quadratic* terms in the expansions (4.1) yields that, again, $r_b - r_a = 2A$.

5 Exact Results for the Moment

The associated stress distribution naturally requires specification of the form of the strain-energy function in order to be determined. The most important functional of this stress distribution is the moment M required to bend the block by an angle α. This moment is given by

$$M = \int_{r_a}^{r_b} T_{\theta\theta} r \mathrm{d}r \tag{5.1}$$

which can be rewritten in terms of the parameter ϵ defined in Eq. (3.15) as follows:

$$\frac{M}{4A^2} = \epsilon^{-2}\int_{\lambda_a}^{\lambda_b} \widetilde{W}(\lambda)\lambda^{-3}\mathrm{d}\lambda + \epsilon^{-1}\widetilde{W}(\lambda_a) \tag{5.2}$$

where λ_a and λ_b have been defined in terms of ϵ through Eq. (3.14). To derive this expression, we made use of Eqs. (2.7a), (3.1b), (3.8a), and (3.15). Note that in Ref. [3] (p. 293), the last term of this expression is missing; note also that the expression of M in terms of an integral in r can be found in Rivlin's original paper [1], see also Kanner and Horgan [4]. It follows trivially from Eqs. (3.8b) and (3.14) that M is an *odd* function of ϵ.

We remark that in this paper, M is the applied moment per unit width of the block: if the block's width is H, then the total applied moment is HM.

The integral term in Eq. (5.2) surprisingly means that explicit relations between the moment and bending angle in terms of elementary functions are difficult to obtain in general. Progress can be made however for some forms of W. For example, Kanner and Horgan [4] compute M for the Mooney–Rivlin material and for the Gent [18] material.

We focus on the following Rivlin expansion of the strain-energy density in the principal invariants of the Cauchy–Green strain tensors [19]

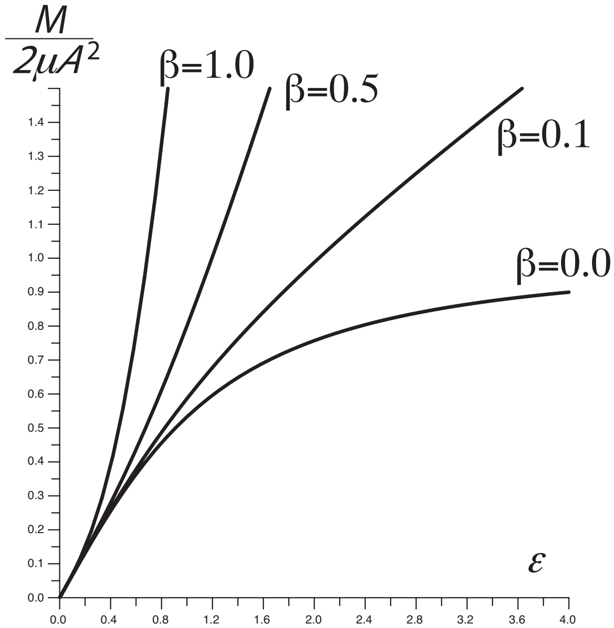

Fig. 3 Variations in the moment M with ϵ, the product of the aspect ratio by the bending angle, for the large bending of an elastic block modeled by Rivlin's strain-energy density (5.3). Exact results in the cases where the nonlinear shear coefficient β is equal to 0.0 (Mooney–Rivlin material), 0.1, 0.5, and 1.0. See Fig. 4 for a comparison with experimental results when β=1.0.

$$W = C_{10}(I-3) + C_{01}(II-3) + C_{20}(I-3)^2 + C_{11}(I-3)(II-3) + C_{02}(II-3)^2 \tag{5.3}$$

where the $C_{ij}(i+j=1,2)$ are constants to be determined from experiments (see Erkamp et al. [16] for example) and

$$I = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad II = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 \tag{5.4}$$

This strain-energy density is readily implemented in most finite element analysis packages. It includes the neo-Hookean and Mooney–Rivlin materials. In the case of plane strain, such as the bending problem considered here $I=II=\lambda^2+\lambda^{-2}+1$ so that W reduces to

$$\widetilde{W} = \frac{1}{2}\mu[(\lambda^2+\lambda^{-2}-2) + \beta(\lambda^2+\lambda^{-2}-2)^2], \tag{5.5}$$

where $\mu=2(C_{10}+C_{10})>0$ is the infinitesimal shear modulus, and $\beta=(C_{20}+C_{11}+C_{02})/\mu>0$ is the nonlinear shear coefficient [11–13]. Substitution into the general moment relation (5.2) then yields the exact, nonlinear relation

$$\frac{M}{2\mu A^2} = (1-4\beta)[\epsilon^{-2}\ln(\sqrt{1+\epsilon^2}-\epsilon) + \epsilon^{-1}\sqrt{1+\epsilon^2}] + \frac{8}{3}\beta\epsilon \tag{5.6}$$

The last term in this expression shows that M grows unbounded with ϵ, except in the special Mooney–Rivlin case for which $\beta=0$ and where the asymptotic value is $M=2\mu A^2$ [4].

In Fig. 3, we plot $M/(2\mu A^2)$ as a function of ϵ for β=0.0, 0.1, 0.5, and 1.0. A noteworthy feature of this plot is that there is a linear response quite far beyond the origin. This is explained in the next section.

6 Approximate Results for the Moment

Expanding the general expression (5.2) for the moment M in a Maclaurin series in ϵ, and noting both that M is odd in ϵ, and Eq. (3.13) yields

$$\frac{M}{A^2} = \frac{1}{3}[\widetilde{W}''(1)]\epsilon + \frac{1}{120}[\widetilde{W}''''(1) + 8\widetilde{W}'''(1) - 3\widetilde{W}''(1)]\epsilon^3 + \ldots \tag{6.1}$$

Recall that only isotropic, incompressible materials are being considered here. Differentiating Eq. (3.6) with respect to λ and evaluation in the reference configuration yields

$$\widetilde{W}''(1) = W_1 + W_2 + W_{11} + W_{22} - 2W_{12} \tag{6.2}$$

where the partial derivatives of W are evaluated at $(\lambda_1,\lambda_2,\lambda_3)=(1,1,1)$. The last of Eqs. (6.1.88) of Ogden [3] then yields

$$\widetilde{W}''(1) = 4\mu \tag{6.3}$$

The dependence of $\widetilde{W}$ on λ can be expressed in the form

$$\widetilde{W}(\lambda) = f(\lambda^2+\lambda^{-2}-2), \tag{6.4}$$

for some function f, say. It then follows that $\widetilde{W}''(1)=8f'(0)$ and $\widetilde{W}'''(1)=-24f'(0)=-3\widetilde{W}''(1)$ or

$$\widetilde{W}'''(1) = -12\mu \tag{6.5}$$

So for isotropic materials the moment relation (6.1) can be written in the form

$$\frac{M}{A^2} = \frac{4}{3}\mu\epsilon + \frac{1}{120}[\widetilde{W}''''(1) - 108\mu]\epsilon^3 + \mathcal{O}(\epsilon^5) \tag{6.6}$$

which explains why the linear regime carries for moderate values of ϵ in Fig. 3.

7 Weak Nonlinear Elasticity

Because we are looking at small but not infinitesimal, elastic effects, we place ourselves in the theory of *weak nonlinear elasticity* [20]. There, the strain-energy density is expanded in terms of

$$I_1 = \mathrm{tr}(\boldsymbol{E}), \quad I_2 = \mathrm{tr}(\boldsymbol{E}^2), \quad I_3 = \mathrm{tr}(\boldsymbol{E}^3) \tag{7.1}$$

where $\boldsymbol{E}$ is the Green–Lagrange strain tensor (with eigenvalues $(\lambda_i^2-1)/2$). For *incompressible* solids, Ogden [21] showed that the expansion of W up to terms which are of order four or less in the Green–Lagrange strain involves only three material constants. In the notation of Hamilton et al. [15], it is written as

$$W = \mu I_2 + \frac{\mathcal{A}}{3}I_3 + \mathcal{D}I_2^2 \tag{7.2}$$

where $\mathcal{A}$ and $\mathcal{D}$ are nonlinear Landau elasticity constants.

The Appendix shows that *at the same order of approximation* in the strains, the Rivlin strain-energy density (5.3) coincides with the fourth-order elasticity expansion (7.2) when

$$\mathcal{A} = -8(C_{10}+2C_{01}), \quad \mathcal{D} = 2(C_{10}+3C_{01}+2C_{20}+2C_{11}+2C_{02}) \tag{7.3}$$

Conversely, we find that β introduced in Eq. (5.5) can be written as

$$\beta = \frac{1}{2}\left(1 + \frac{\mathcal{A}/2+\mathcal{D}}{\mu}\right) \tag{7.4}$$

in agreement with Zabolotzkaya et al. [11].

Consequently the general moment-ϵ relation (6.6) for fourth-order elasticity has the form

$$\frac{M}{A^2} = \frac{4}{3}\mu\epsilon + \frac{2}{5}(\mu+\mathcal{A}+2\mathcal{D})\epsilon^3 + \mathcal{O}(\epsilon^5) \tag{7.5}$$

or, in nondimensional form,

$$\frac{M}{\mu A^2} = \frac{4}{3}\epsilon + \frac{2}{5}(4\beta - 1)\epsilon^3 + \mathcal{O}(\epsilon^5) \tag{7.6}$$

We remark that this expansion is in agreement with the Maclaurin series derived from the exact expression (5.6), see Rivlin [1]. We also note that the *fourth-order elasticity is necessary and sufficient* to express the onset of nonlinearity present in the coefficient of ϵ^3 in Eq. (6.6): third-order elasticity cannot account for all the components of that coefficient, and we checked that fifth-order constants do not appear in it (these calculations are not reproduced here).

Finally, we check that the expansion (7.6) is consistent with classic elastic theory, which tells us that the total flexural moment required to achieve a curvature κ of a plate with width H and thickness $2A$ is

$$HM = EI\kappa, \tag{7.7}$$

where E is Young's modulus and I is the moment of inertia of cross-sectional area. Here the block is rectangular and $I = H(2A)^3/12$. Also, the curvature is $\kappa = \alpha/L$, see Sec. 4. Moreover, it is known that in classic plane strain theory, $E = 2\mu(1+\nu)/(1-\nu^2)$, where ν is Poisson's ratio. As we are dealing only with incompressible solids, $\nu = 1/2$, and we find that the linear term in Eq. (7.6) or in Eq. (6.6) (or the term obtained from a linear expansion in ϵ of Eq. (5.6)) is indeed given by Eq. (7.7).

8 Experimental Results

We conducted bending experiments on several strips of elastomers, using a Tinius Olsen bending stiffness tester. That tester meets the requirements of the ASTM standard test method E855 [17]. We used strips, which were about 4.5 mm thick. The tester imposes a moment at two points of the strip separated by half-an-inch. Hence the aspect ratio of the strips with respect to the bending experiments, was $A/L = 4.5/12.7 \simeq 0.35$. We bent the samples by small, moderate, and large bending angles but noticed that at large angles, pinching (and perhaps also wrinkling [8]) took place on the inner face of the bent strip. Consequently we only retained the data up to an angle of 60 deg, which gives the following range for the expansion parameter: $0 \leq \epsilon \leq 0.37$.

In Fig. 4, we show the results of one representative experiment, for a strip of polyurethane elastomer, shore hardness 40A. On the vertical axis, the variable is a nondimensional measure of the moment, M/M_m, where M_m is the maximum bending moment of the tester's pendulum (see ASTM standard test method E855 [17] for details); the actual value of M_m is irrelevant, as we are only interested in measuring the nondimensional parameter β. On the horizontal axis, the variable is ϵ. The circles represent the recorded experimental data (16 measurements in the 0–60 deg range for the angle of bending, see Table 1). The straight dashed line corresponds to the fitting with linear elasticity theory ($\beta = 0$, Eq. (7.7)), and the full thick plot corresponds to the fitting with fourth-order elasticity theory (7.6). Only one parameter (β) is to be determined from the cubic relation (7.6), which ensures the existence and uniqueness of the fitting parameter β. We obtained a good agreement when $\beta = 1.0$.

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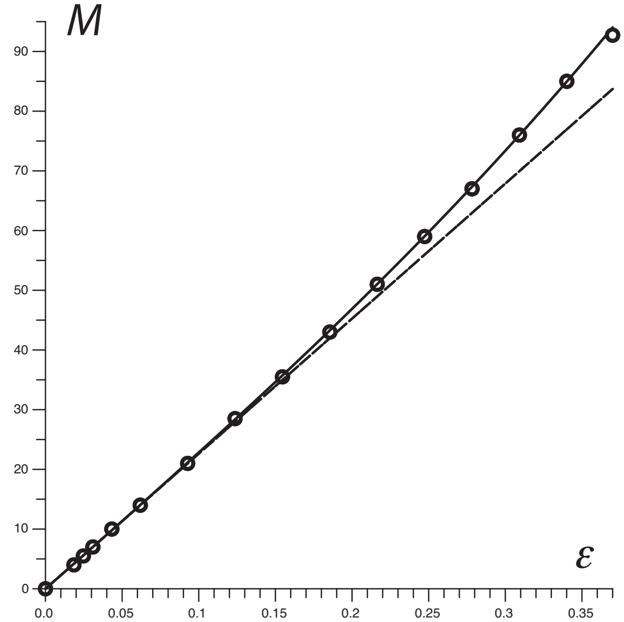

Fig. 4 Bending of a strip of polyurethane: variation in the nondimensional measure of the bending moment M/M_m with ϵ, the product of the strip's aspect ratio by the angle of bending. Circles: experimental data; dashed straight line: linear elasticity, Eq. (7.7); and full line plot: fitting of third- and fourth-order elasticity effects with the data by adjusting the nonlinear parameter β to the value 1.0 in Eq. (7.6).

Appendix: Correspondence Between Exact Nonlinear Elasticity and Weakly Nonlinear Elasticity

In the exact (finite) theory of nonlinear elasticity, there are no restrictions to impose on the magnitude of the strain. Often the strain-energy density W is written in terms of the first three principal invariants of the Cauchy–Green right strain tensor $\boldsymbol{C}$

$$I = \text{tr}(\boldsymbol{C}), \quad II = \frac{1}{2}[(\text{tr}\boldsymbol{C})^2 - \text{tr}(\boldsymbol{C}^2)], \quad III = \det \boldsymbol{C} \tag{A1}$$

For incompressible solids, $III = 1$ at all times and $W = W(I, II)$ only.

In the weakly nonlinear theory of elasticity, W is expanded up to a certain order in a certain measure of strain, and all higher-order terms are neglected. Often the Green–Lagrange strain tensor $\boldsymbol{E} = (\boldsymbol{C} - \boldsymbol{I})/2$ is favored and the expansion is made in terms of the quantities I_1, I_2, and I_3 defined in Eq. (7.1).

There exist of course connections between the two theories. For instance, Rivlin and Saunders [22] show that the Mooney strain-energy density of exact nonlinear incompressible elasticity

$$W = C_{10}(I - 3) + C_{01}(II - 3), \tag{A2}$$

coincides, at the same order of approximation with the general weakly nonlinear third-order elasticity expansion,

$$W = \mu I_2 + \frac{\mathcal{A}}{3} I_3 \tag{A3}$$

The connections between the material constants C_{01}, C_{10}, μ, and $\mathcal{A}$ are

$$\mu = 2(C_{10} + C_{01}), \quad \mathcal{A} = -8(C_{10} + 2C_{01}) \tag{A4}$$

or conversely

$$C_{10} = \frac{1}{2}\left(2\mu + \frac{\mathcal{A}}{4}\right), \quad C_{01} = -\frac{1}{2}\left(\mu + \frac{\mathcal{A}}{4}\right) \tag{A5}$$

see Goriely et al. [23], where there is a misprint. Now we show how Rivlin's strain-energy density (5.3) is connected to the fourth-order elasticity expansion (7.2) for incompressible solids.

The general relations between I, II, and III and I_1, I_2, and I_3 are well-known and straight-forward to derive

$$I = 3 + 2I_1$$

$$II = 3 + 4I_1 - 2I_2 + 2I_1^2$$

$$III = 1 + 2I_1 + 2I_1^2 - 2I_2 + \tfrac{4}{3}I_1^3 - 4I_1I_2 + \tfrac{8}{3}I_3 \tag{A6}$$

For incompressible solids, $III=1$ is enforced at all times, and so

$$I_1 = -I_1^2 + I_2 - \tfrac{2}{3}I_1^3 + 2I_1I_2 - \tfrac{4}{3}I_3 \tag{A7}$$

showing that I_1 is at least of order 2. Squaring gives

$$I_1^2 = I_2^2 + \text{HOT} \tag{A8}$$

where "HOT" is the acronym for "higher-order terms" (here, higher than fourth-order terms). Multiplying Eq. (A7) by I_2 yields $I_1I_2=I_2^2-I_1^2I_2+\text{HOT}$ or using Eq. (A8)

$$I_1I_2 = I_2^2 + \text{HOT} \tag{A9}$$

Substituting Eqs. (A8) and (A9) into Eq. (A7) gives [15,14]

$$I_1 = I_2 - \tfrac{4}{3}I_3 + I_2^2 + \text{HOT} \tag{A10}$$

Hence, the relations (A6) reduce, for incompressible solids, to

$$I - 3 = 2I_2 - \tfrac{8}{3}I_3 + 2I_2^2 + \text{HOT}$$

$$II - 3 = 2I_2 - \tfrac{16}{3}I_3 + 6I_2^2 + \text{HOT}$$

$$(I-3)^2 = (I-3)(II-3) = (II-3)^2 = 4I_2^2 + \text{HOT} \tag{A11}$$

and Rivlin's strain-energy density (5.3) reduces to

$$W = 2(C_{10}+C_{01})I_2 - \frac{8}{3}(C_{10}+2C_{01})I_3 + 2(C_{10}+3C_{01}+2C_{20}+2C_{11}+2C_{02})I_2^2. \tag{A12}$$

Clearly, it coincides with the fourth-order elasticity expansion (7.2) with connections (7.3).

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