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On stress-dependent elastic moduli and wave speeds

M. DESTRADE
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, University Road, Galway, Ireland
michel.destrade@nuigalway.ie

AND

R. W. OGDEN*
School of Engineering, University of Aberdeen, King's College, Aberdeen AB24 3UE, UK
*Corresponding author: r.ogden@abdn.ac.uk

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On the basis of the general non-linear theory of a hyperelastic material with initial stress, initially without consideration of the origin of the initial stress, we determine explicit expressions for the stress-dependent tensor of incremental elastic moduli. In considering three special cases of initial stress within the general framework, namely hydrostatic stress, uniaxial stress and planar shear stress, we then elucidate in general form the dependence of various elastic moduli on the initial stress. In each case, the effect of initial stress on the wave speed of homogeneous plane waves is studied and it is shown how various special theories from the earlier literature fit within the general framework. We then consider the situation in which the initial stress is a pre-stress associated with a finite deformation and, in particular, we discuss the specialization to the second-order theory of elasticity and highlight connections between several classical approaches to the topic, again with special reference to the influence of higher-order terms on the speed of homogeneous plane waves. Some discrepancies arising in the earlier literature are noted.

Keywords: elastic moduli; isotropic stress; initial stress; invariants; plane waves.

1. Introduction

Residual stresses in solids, i.e. stresses that are present in the absence of load (body forces and surface tractions), can have a very significant effect on the mechanical behaviour of the structures in which they reside. This is the case for materials as diverse as hard engineering and geological materials and soft solids such as gels and biological tissues. Equally, stresses that are generated due to applied loads, associated with finite deformations and commonly referred to as pre-stresses, have a significant effect on subsequent material response, leading to very different results compared with the situation in which there is no applied load. Residual stresses and pre-stresses are examples of initial stresses but are different in nature in the sense that residual stresses are necessarily non-homogeneous, while pre-stresses may be either homogeneous or non-homogeneous. Also, pre-stresses are usually associated with an elastic pre-deformation, while residual stresses can result from processes such as manufacturing, plastic deformation, growth and remodelling, for example. In either case, it is important to be able to analyse the effect of the initial stress on the properties of the material and on its mechanical response. In this paper, we are concerned primarily with the effect of initial stress on the material properties in general, and on elastic 'constants' in some specific cases, as well as its effect on the speeds of propagation of homogeneous plane waves. The study is conducted with a view to the non-destructive evaluation of initially stressed solids, which are ubiquitous in Nature and Engineering. For this purpose, the initial stress is included in the constitutive description of the material without reference (initially) to any finite deformation with which it may be associated, and we emphasize that in general, the initial stress appears in a highly non-linear form.

The origins of an elasticity theory including initial stress can be traced back as far as to the works of Cauchy (1829), according to Truesdell (1966). Notable early contributors to the subject include Poincaré (1892), Hadamard (1903), Rayleigh (1906), Brillouin (1925) and Love (1927).

In the context of the modern linear theory of elasticity, the effect of initial stress was first examined in the work of Biot for static problems (Biot, 1939) and also for wave propagation problems (Biot, 1940a); see also his monograph (Biot, 1965). Here, for later reference, we write the components of Biot's elasticity tensor as $\mathcal{B}_{piqj}$ with respect to a Cartesian coordinate system (x_1, x_2, x_3). In general, these enjoy the minor symmetries

$$\mathcal{B}_{piqj} = \mathcal{B}_{ipqj} = \mathcal{B}_{pijq}, \tag{1.1}$$

but when there exists a strain-energy function, there is also the major symmetry connection

$$\mathcal{B}_{piqj} - \mathcal{B}_{qjpi} = \delta_{ip}\tau_{jq} - \delta_{jq}\tau_{ip}, \tag{1.2}$$

where τ_{ij} are the components of the initial Cauchy stress and δ_{ij} is the Kronecker delta. For infinitesimal strains e_{ij}, with $e_{ij} = (u_{i,j} + u_{j,i})/2$, where $u_{i,j} = \partial u_i/\partial x_j$ and u_i are the components of the displacement vector, the associated Cauchy stress, additional to the initial stress, is given by $\sigma_{pi} = \mathcal{B}_{piqj}u_{j,q}$. Biot left the dependence of $\mathcal{B}_{piqj}$ on the initial stress unspecified for most of his theoretical development and he was not concerned with the source of the initial stress. In this sense, his theory may be referred to as the *general linear theory of elasticity with initial stress*. Biot was more specific in particular cases, where he considered both isotropic and planar orthotropic specializations and an initial stress due to hydrostatic pressure, uniaxial compression or gravity. In particular, for his isotropic model, $\mathcal{B}_{piqj}$ may be written in the form

$$\mathcal{B}_{piqj} = \mu_0(\delta_{ij}\delta_{pq} + \delta_{qi}\delta_{pj}) + \lambda_0\delta_{pi}\delta_{qj} - \delta_{qj}\tau_{pi}, \tag{1.3}$$

where μ_0 and λ_0 are the notations that we shall use in this paper for the classical Lamé moduli of linear isotropic elasticity. This form did not appear explicitly in Biot's work, as far as the authors are aware, but may be deduced from the plane strain expressions in (8.31e) of Biot's book (Biot, 1965). Biot did in fact acknowledge that in general, the elastic response in the presence of initial stress is not isotropic, but he adopted an isotropic constitutive description for simplicity. Note that (1.3) satisfies the conditions (1.2).

The works of Biot have formed the basis for many contributions to the literature, particularly relating to wave propagation problems in the geophysical context, which was the original context in which the theory was developed by Biot (1940a). Other contributions to the analysis of initial stress, more specifically residual stress, have appeared in a series of papers by Hoger and co-workers, including Hoger (1985, 1986, 1993a,b, 1996), Johnson & Hoger (1993), some of which are concerned with the combined effect of finite deformation and residual stress and in papers by Man & Lu (1987), Man (1998) and Saravanan (2008), for example. More recently, a general theory of non-linear hyperelasticity for an initially stressed solid has been developed by Shams *et al.* (2011) and Ogden & Singh (2011), the latter being focused on fibre reinforced materials. We also mention the paper by Bažant (1971), who detailed connections between several earlier formulations of linear elasticity with initial stress and their implications for the stability of elastic bodies.

We distinguish between the above approach and that concerned with the effect of pre-stress that is associated with a finite deformation, a subject that has attracted many contributions, primarily concerned with the effect of the finite deformation on the propagation of small amplitude elastic waves and associated static bifurcation problems. This is commonly known as the *theory of incremental (or small) deformations superimposed on a finite deformation*. We shall not discuss this extensive topic in detail but refer to Ogden (1984) and Ogden (2007), for example, for pointers to the literature.

A special case of the theory of finite elastic deformations in which the strains are small but the linear theory is no longer adequate is sometimes referred to as second-order elasticity. In this theory, the strain-energy function is expanded to the third order in some suitable measure of strain and the stress is second order in the strain. Most commonly, it is the Green (or Green–Lagrange) strain tensor that is used and for an isotropic material, the strain energy is expressed in terms of invariants of the strain. The first such contribution appears to be that of Brillouin (1925), although this has not always been acknowledged appropriately. Equivalent formulations were developed later by Landau & Rumer (1937), Murnaghan (1937), Biot (1940b), Toupin & Bernstein (1961) and Hayes & Rivlin (1961); see also the books by Brillouin (1946), Biot (1965) and Landau & Lifshitz (1986). There is also a detailed historical discussion in the volume by Truesdell & Noll (1965), in particular in Section 66 therein. An important objective within these works was to determine the correction to the speeds of waves due to the non-linear terms in the stress. In particular, various formulas were found that highlighted the effect of an initial hydrostatic pressure (associated with a pure dilatation) or uniaxial compression on the speeds of longitudinal and transverse waves. Of other works dealing with wave speeds based on this weakly non-linear theory, we mention those of Hughes & Kelly (1953), who obtained experimental results for longitudinal and transverse wave speeds in polystyrene, iron and pyrex for separate initial stresses corresponding to hydrostatic pressure and simple compression and related their results to formulas based on Murnaghan's second-order theory, and Thurston & Brugger (1964) who obtained expressions for the second-order corrections to wave speeds in cubic crystals. Papers by Birch (1938) and Tang (1967) also made use of the second-order theory but failed to include the second-order constants in their expressions for the wave speeds (note that the second-order constants are sometimes referred to as third-order constants). We discuss this and other deficiencies of the latter paper in Sections 6.1, 6.3 and 7.3.

For several researchers, it seemed important to show that all elastic materials would behave similarly under an initial stress. Hence, Biot's incremental moduli (1.3) change from one material to another with changes in the values of λ_0 and μ_0, but the effect of the pre-stress $\boldsymbol{\tau}$ remains the same across all solids. Similarly, according to Lazarus (1949), Love (1927) showed that under external pressure P, the (linear) elastic constants c_{44} and c_{12} (Voigt notation) of a certain class of solid are linked approximately by the 'Cauchy relation' $c_{44} = c_{12} - 2P$. This type of behaviour under pre-stress would in turn lead to a corresponding effect on the shift in speed experienced by an acoustic wave (for instance a wave should always travel faster in a pressurized isotropic solid than in its unstressed counterpart). However intuitive this expectation might be, it is not supported by experimental facts, as confirmed by the data shown in Tables 1 and 2, which show that the wave speed can increase or decrease with pressure, depending on the material.

The purpose of the present paper is to draw together and highlight connections between some of the historical results within a common and fairly general framework based on the development of Shams *et al.* (2011) concerned with the constitutive law of a hyperelastic material with initial stress. In particular, we examine how the elasticity tensor depends non-linearly on initial stress, with emphasis on the important special cases of hydrostatic initial stress, uniaxial initial stress and initial shear stress.

In Section 2, we summarize the basic equations for an elastic material for which the strain-energy function depends on an initial stress as well as the deformation from the initially stressed reference

TABLE 1 *Initial variation of the squared wave speeds for several solids under hydrostatic pressure P, as collected by Johnson et al. (1994): ρ_r is the mass density at $P = 0$; υ_{T} and υ_{L} are the speeds of the transverse and longitudinal waves, respectively*

Solid	$\left.\dfrac{\mathrm{d}}{\mathrm{d}P}(\rho_r \upsilon_{\mathrm{T}}^2)\right\|_{P=0}$	$\left.\dfrac{\mathrm{d}}{\mathrm{d}P}(\rho_r \upsilon_{\mathrm{L}}^2)\right\|_{P=0}$
Alumina	1.12	4.46
Aluminum	2.92	12.4
Armco-Iron	5.7	9.3
Fused silica	−1.42	−4.32
Gold	0.90	6.4
Magnesium	1.47	6.89
Molybdenum	1.05	3.48
Nickel-steel	1.55	2.84
Niobium	0.29	6.18
PMMA	3.0	15.0
Polystryrene	1.57	11.6
Pyrex	−2.84	−8.6
Steel (Hecla)	1.46	7.45
Tungsten	0.70	4.58

TABLE 2 *Initial variation of the speeds of transverse waves for several solids under uniaxial strain ϵ: υ_{12} and υ_{21} are the speeds waves travelling in the direction of tension and orthogonal to the direction of tension, respectively; υ_{12}^0 and υ_{21}^0 are their values at $\epsilon = 0$. The first row of data is from experiments on a sample of rail steel (Egle & Bray, 1976) and the other rows are from experiments on soft solids with different compositions (Gennisson et al., 2007)*

Solid	$\left.\dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\dfrac{\upsilon_{12}}{\upsilon_{12}^0}\right)\right\|_{\epsilon=0}$	$\left.\dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\dfrac{\upsilon_{21}}{\upsilon_{21}^0}\right)\right\|_{\epsilon=0}$
Rail steel 1	−0.15	−1.50
Agar-Gelatine 1	0.84	−0.92
Agar-Gelatine 4	4.53	2.26
Polyvinyl acetate 1	0.71	−0.47
Polyvinyl acetate 3	1.69	0.35

configuration. In particular, we express the strain energy as a function of combined invariants of the initial stress and deformation and give expressions for the nominal and Cauchy stress tensors. Next, in Section 3, we derive the equations of motion for small displacements from a homogeneously deformed configuration when the initial stress is uniform, which leads to the need for expressions for the elasticity tensor. Such expressions are given in Section 4 but now specialized to the undeformed (but initially stressed) reference configuration. Several special cases are considered in which the initial stress is either purely isotropic, uniaxial, or a planar shear stress. These formulas are then used in Section 5 to define relevant elastic moduli that depend on the initial stress in question and to make that dependence explicit. The moduli include stress-dependent Lamé moduli in the case of isotropic initial stress, Poisson's ratios and Young's moduli for uniaxial initial stress, and planar Poisson's ratios and Young's moduli for planar initial shear stress.

In Section 6, the results are applied to infinitesimal wave propagation and related to some known results as special cases. Section 7 then considers the deformation of an isotropic elastic material from a stress-free reference configuration in order to make contact with results in the preceding sections by considering, in particular, a pure dilatation and a deformation corresponding to simple tension. We then focus on the specialization to second-order elasticity in which the strain energy is approximated as a third-order expansion in the Green strain tensor in order to highlight connections with the theory of elasticity with initial stress herein and to draw together various contributions from the literature that date back to the work of Brillouin (1925), with particular reference to expressions for longitudinal and transverse wave speeds.

2. Elasticity in the presence of initial stress

We consider an elastic body that is subject to an initial (Cauchy) stress $\boldsymbol{\tau}$ in some *reference configuration*, which we denote by $\mathcal{B}_{\mathrm{r}}$. In the absence of intrinsic couple stresses, $\boldsymbol{\tau}$ is symmetric. Let $\mathbf{X}$ be the position vector of a material point in $\mathcal{B}_{\mathrm{r}}$ and let Grad and Div denote the gradient and divergence operators with respect to $\mathbf{X}$. If there are no body forces, then $\boldsymbol{\tau}$ must satisfy the equilibrium equation $\operatorname{Div}\boldsymbol{\tau} = \mathbf{0}$. For the most part, we shall not be concerned with how the initial stress arises, but in Section 7, we shall relate $\boldsymbol{\tau}$ to an underlying finite deformation and $\boldsymbol{\tau}$ is then considered to be a *pre-stress*, which requires appropriate tractions on the boundary $\partial\mathcal{B}_{\mathrm{r}}$ of $\mathcal{B}_{\mathrm{r}}$. However, if the traction on the boundary $\partial\mathcal{B}_{\mathrm{r}}$ of $\mathcal{B}_{\mathrm{r}}$ vanishes pointwise, then $\boldsymbol{\tau}$ is referred to as a *residual stress*. A residual stress is necessarily non-uniform (Hoger, 1985; Ogden, 2003) and in general is not associated with a deformation from a stress-free configuration.

Next, we consider the body to be subject to a finite elastic deformation from $\mathcal{B}_{\mathrm{r}}$ into a new configuration $\mathcal{B}$ with boundary $\partial\mathcal{B}$ so that the material point $\mathbf{X}$ takes up the position $\mathbf{x}$ in $\mathcal{B}$ given by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where the vector function $\boldsymbol{\chi}$ defines the deformation for $\mathbf{X} \in \mathcal{B}_{\mathrm{r}}$. The *deformation* $\boldsymbol{\chi}$ is required to be a bijection and to possess appropriate regularity properties, which we do not need to specify here. The deformation gradient tensor, denoted $\mathbf{F}$, is defined by $\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}$ from which are formed the left and right Cauchy–Green deformation tensors, defined by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\top}, \quad \mathbf{C} = \mathbf{F}^{\top}\mathbf{F}, \tag{2.1}$$

respectively.

We denote by $\boldsymbol{\sigma}$ the Cauchy stress tensor in the configuration $\mathcal{B}$ and we suppose that there are no body forces, so that the equilibrium equation $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$ holds. We shall also make use of the nominal and the second Piola–Kirchhoff stress tensors, denoted $\mathbf{S}$ and $\mathbf{T}$, respectively, which are related to $\boldsymbol{\sigma}$ and each other by

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma} = \mathbf{T}\mathbf{F}^{\top}, \quad \mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\top} = \mathbf{S}\mathbf{F}^{-\top}, \tag{2.2}$$

where $J = \det\mathbf{F} > 0$. The nominal stress $\mathbf{S}$ satisfies the equilibrium equation

$$\operatorname{Div}\mathbf{S} = \mathbf{0}. \tag{2.3}$$

In the absence of intrinsic couples, $\boldsymbol{\sigma}$, and hence $\mathbf{T}$, is symmetric, while in general, $\mathbf{S}$ is not symmetric and satisfies

$$\mathbf{F}\mathbf{S} = \mathbf{S}^{\top}\mathbf{F}^{\top}. \tag{2.4}$$

We now consider the elastic properties of the material to be characterized in terms of a strain-energy function, defined per unit volume in $\mathcal{B}_r$, which we denote by W. We write

$$W = W(\mathbf{F}, \boldsymbol{\tau}) \tag{2.5}$$

to reflect the dependence not only on the deformation gradient but also on the initial stress. By objectivity, W depends on $\mathbf{F}$ only through $\mathbf{C} = \mathbf{F}^\top\mathbf{F}$, but it is convenient to retain the functional dependence indicated in (2.5). In general, the presence of the initial stress will generate anisotropy in the material response relative to $\mathcal{B}_r$ and $\boldsymbol{\tau}$ has a role similar to that of a structure tensor associated with a preferred direction in an anisotropic material. An exception to this arises if $\boldsymbol{\tau}$ is an isotropic stress. If $\boldsymbol{\tau}$ is non-uniform, then the material is necessarily inhomogeneous, but if $\boldsymbol{\tau}$ is independent of $\mathbf{X}$, the material is homogeneous unless its properties depend separately on $\mathbf{X}$. In the present paper, we shall consider $\boldsymbol{\tau}$ to be uniform and the material to be homogeneous.

We shall not consider internal constraints such as incompressibility in which case the nominal stress is given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}, \boldsymbol{\tau}), \tag{2.6}$$

and the Cauchy and second Piola–Kirchhoff stresses can be obtained from (2.2). When evaluated in $\mathcal{B}_r$, (2.6) reduces to

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}, \boldsymbol{\tau}), \tag{2.7}$$

where $\mathbf{I}$ is the identity tensor. Equation (2.7) imposes a restriction on the admissible forms of strain-energy function for an initially stressed elastic material.

2.1 *Invariant representation of the strain energy and stresses*

The strain-energy function W depends on $\mathbf{C}$ and $\boldsymbol{\tau}$, both of which are independent of rotations in the deformed configuration $\mathcal{B}$. Thus, W is automatically objective. If the material possesses no intrinsic anisotropy relative to $\mathcal{B}_r$, so that it would be isotropic relative to $\mathcal{B}_r$ in the absence of initial stress, then W is an isotropic function of $\mathbf{C}$ and $\boldsymbol{\tau}$, i.e.

$$W(\mathbf{QCQ}^\top, \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^\top) = W(\mathbf{C}, \boldsymbol{\tau}) \quad \text{for all orthogonal } \mathbf{Q}, \tag{2.8}$$

and it can be expressed as a function of the invariants of $\mathbf{C}$ and $\boldsymbol{\tau}$. We list a possible (and complete) set of independent invariants as

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)], \quad I_3 = \det\mathbf{C}, \tag{2.9}$$

$$\operatorname{tr}\boldsymbol{\tau}, \quad \operatorname{tr}(\boldsymbol{\tau}^2), \quad \operatorname{tr}(\boldsymbol{\tau}^3), \tag{2.10}$$

$$I_6 = \operatorname{tr}(\boldsymbol{\tau}\mathbf{C}), \quad I_7 = \operatorname{tr}(\boldsymbol{\tau}\mathbf{C}^2), \quad I_8 = \operatorname{tr}(\boldsymbol{\tau}^2\mathbf{C}), \quad I_9 = \operatorname{tr}(\boldsymbol{\tau}^2\mathbf{C}^2), \tag{2.11}$$

where we have used the standard notation I_1, I_2, I_3 for the principal invariants of $\mathbf{C}$ and followed the notation $I_6, \dots, I_9$ adopted by Shams *et al.* (2011) for the combined invariants of $\mathbf{C}$ and $\boldsymbol{\tau}$. In the reference configuration $\mathcal{B}_r$, these reduce to

$$I_1 = I_2 = 3, \quad I_3 = 1, \quad I_6 = I_7 = \operatorname{tr}\boldsymbol{\tau}, \quad I_8 = I_9 = \operatorname{tr}(\boldsymbol{\tau}^2). \tag{2.12}$$

For full discussion of invariants of tensors, we refer to Spencer (1971) and Zheng (1994). Here, there are 10 independent invariants of $\mathbf{C}$ and $\boldsymbol{\tau}$ in general, a number that may be reduced in a 2D specialization or for specific simple deformations and/or initial stresses. We have not attributed notations to the invariants (2.10) since they are independent of the deformation and do not contribute explicitly to expressions for the stresses. However, W may depend on (2.10) implicitly, but we do not list them in the functional dependence and we write $W = W(I_1, I_2, I_3, I_6, I_7, I_8, I_9)$, retaining the notation W, which is used severally for the $(\mathbf{F}, \boldsymbol{\tau})$, $(\mathbf{C}, \boldsymbol{\tau})$ and $(I_1, I_2, I_3, I_6, I_7, I_8, I_9)$ arguments.

From (2.6), the nominal stress tensor may be expanded as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} = \sum_{i \in \mathcal{I}} W_i \frac{\partial I_i}{\partial \mathbf{F}}, \tag{2.13}$$

where we have used the shorthand notation $W_i = \partial W / \partial I_i$, $i \in \mathcal{I}$, and $\mathcal{I}$ is the index set $\{1, 2, 3, 6, 7, 8, 9\}$. We emphasize that although their derivatives with respect to $\mathbf{F}$ vanish, the invariants (2.10) are included implicitly in the functional dependence of W in general. The required expressions for $\partial I_i / \partial \mathbf{F}$ were given in Appendix A of Shams *et al.* (2011) and are not repeated here explicitly but used implicitly. The resulting expression for the Cauchy stress $\boldsymbol{\sigma}$ is obtained from (2.2) in the form

$$\begin{aligned} J\boldsymbol{\sigma} &= 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + 2I_3W_3\mathbf{I} + 2W_6\boldsymbol{\Sigma} \\ &\quad + 2W_7(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}) + 2W_8\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma} + 2W_9(\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}), \end{aligned} \tag{2.14}$$

wherein we have introduced the notation $\boldsymbol{\Sigma} = \mathbf{F}\boldsymbol{\tau}\mathbf{F}^\top$ for the push forward of $\boldsymbol{\tau}$ from $\mathcal{B}_\mathrm{r}$ to $\mathcal{B}$, and $\mathbf{B} = \mathbf{F}\mathbf{F}^\top$ is the left Cauchy–Green tensor.

When (2.14) is evaluated in the reference configuration, it reduces to

$$\boldsymbol{\tau} = 2(W_1 + 2W_2 + W_3)\mathbf{I} + 2(W_6 + 2W_7)\boldsymbol{\tau} + 2(W_8 + 2W_9)\boldsymbol{\tau}^2, \tag{2.15}$$

where $W_i, i \in \mathcal{I}$, are evaluated for the invariants given by (2.12). Equation (2.15) is the specialization of (2.7) for the invariant form of W. As in Shams *et al.* (2011), we deduce that

$$W_1 + 2W_2 + W_3 = 0, \quad 2(W_6 + 2W_7) = 1, \quad W_8 + 2W_9 = 0 \tag{2.16}$$

in $\mathcal{B}_\mathrm{r}$.

3. Incremental motions

We now consider the static finite deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ that defines the equilibrium configuration $\mathcal{B}$ to be followed by a superimposed incremental motion $\dot{\mathbf{x}}(\mathbf{X}, t)$, where t is time. A superposed dot signifies an incremental quantity and the resulting incremental equations are linearized in the increments, which are considered appropriately ‘small’. Thus, $\dot{\mathbf{x}}$ represents a small displacement from $\mathbf{x}$. We shall also write the displacement in Eulerian form as $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, noting that $\dot{\mathbf{x}}(\mathbf{X}, t) = \mathbf{u}(\boldsymbol{\chi}(\mathbf{X}), t)$. The corresponding increments in the deformation gradient $\mathbf{F}$ and $J = \det \mathbf{F}$ are given by the standard formulas

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \quad \dot{J} = J \operatorname{tr} \mathbf{L}, \tag{3.1}$$

where $\mathbf{L} = \operatorname{grad} \mathbf{u}$ is the displacement gradient.

The (linearized) incremental nominal stress is written

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}}, \tag{3.2}$$

where

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}}, \quad \mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}} \tag{3.3}$$

is the fourth-order *elasticity tensor* and, in component form, $\boldsymbol{\mathcal{A}}\dot{\mathbf{F}} \equiv \mathcal{A}_{\alpha i \beta j} \dot{F}_{j\beta}$ defines the product used in (3.2). The usual summation convention for repeated indices is adopted here and henceforth. For full discussion of the theory of incremental deformations and motions superimposed on a finite deformation, we refer to Ogden (1984, 2007), for example.

By taking the increments of the connections $J\boldsymbol{\sigma} = \mathbf{FS}$ and $\mathbf{S} = \mathbf{TF}^\top$ from (2.2) we obtain, after a little rearrangement,

$$\dot{\mathbf{S}}_0 \equiv J^{-1}\mathbf{F}\dot{\mathbf{S}} = \dot{\boldsymbol{\sigma}} + (\mathrm{tr}\,\mathbf{L})\boldsymbol{\sigma} - \mathbf{L}\boldsymbol{\sigma}, \tag{3.4}$$

$$\dot{\mathbf{T}}_0 \equiv J^{-1}\mathbf{F}\dot{\mathbf{T}}\mathbf{F}^\top = \dot{\mathbf{S}}_0 - \boldsymbol{\sigma}\mathbf{L}^\top, \tag{3.5}$$

wherein the notations $\dot{\mathbf{S}}_0$ and $\dot{\mathbf{T}}_0$ are defined. These are the updated forms of $\dot{\mathbf{S}}$ and $\dot{\mathbf{T}}$, respectively, referred to deformed configuration and otherwise know as their 'push forward' forms. The corresponding push forward $\boldsymbol{\mathcal{A}}_0$ of the elasticity tensor is such that $\dot{\mathbf{S}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L}$. It then follows from the symmetry of $\boldsymbol{\sigma}$ and its increment that $\dot{\mathbf{S}}_0 + \mathbf{L}\boldsymbol{\sigma}$ is symmetric and hence

$$\boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbf{L}\boldsymbol{\sigma} = (\boldsymbol{\mathcal{A}}_0\mathbf{L})^\top + \boldsymbol{\sigma}\mathbf{L}^\top. \tag{3.6}$$

In component form $\boldsymbol{\mathcal{A}}_0$ is related to $\boldsymbol{\mathcal{A}}$ via

$$J\mathcal{A}_{0piqj} = F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i \beta j}. \tag{3.7}$$

Note that as well as possessing the major symmetry $\mathcal{A}_{0piqj} = \mathcal{A}_{0qjpi}$, which follows from (3.3) and (3.7), $\boldsymbol{\mathcal{A}}_0$ has the property

$$\mathcal{A}_{0piqj} + \delta_{jp}\sigma_{iq} = \mathcal{A}_{0ipqj} + \delta_{ij}\sigma_{pq}, \tag{3.8}$$

which can be deduced from (3.6).

We assume that there are no body forces. Then, the incremental motion is governed by the equation

$$\mathrm{Div}\,\dot{\mathbf{S}} = \rho_\mathrm{r}\mathbf{x}_{,tt}, \tag{3.9}$$

where ρ_r is the mass density in $\mathcal{B}_\mathrm{r}$ and a subscript t following a comma signifies the material time derivative, i.e. the time derivative at fixed $\mathbf{X}$, so that $\mathbf{x}_{,t} = \mathbf{u}_{,t}$ is the particle velocity and $\mathbf{x}_{,tt} = \mathbf{u}_{,tt}$ the acceleration.

Equation (3.9) may be updated (i.e. pushed forward) to the configuration $\mathcal{B}$ by writing it in terms of $\dot{\mathbf{S}}_0$ and $\mathbf{u}$, which yields

$$\mathrm{div}\,\dot{\mathbf{S}}_0 = \rho\mathbf{u}_{,tt}, \tag{3.10}$$

where $\rho = \rho_\mathrm{r}J^{-1}$ is the mass density in $\mathcal{B}$, or, in (Cartesian) component form,

$$(\mathcal{A}_{0piqj}u_{j,q})_{,p} = \rho u_{i,tt}. \tag{3.11}$$

Henceforth, in this paper, we assume that the initial stress $\boldsymbol{\tau}$, the underlying deformation $\mathbf{F}$ and the material properties are homogeneous, so that $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}_0$ are independent of $\mathbf{X}$. The equation of motion (3.11) then becomes

$$\mathcal{A}_{0piqj}u_{j,pq} = \rho u_{i,tt}. \tag{3.12}$$

This will be used in Section 6 in discussion of the propagation of homogeneous plane waves, but before proceeding to that analysis, we obtain explicit expressions for the dependence of the components of the elasticity tensor and of various elastic moduli on the initial stress based on the invariants of the right Cauchy–Green deformation tensor $\mathbf{C}$ and the initial stress tensor $\boldsymbol{\tau}$ discussed in Section 2.1.

4. Expressions for the elasticity tensor

The elasticity tensor $\boldsymbol{\mathcal{A}}$ in (3.3) may be expanded in terms of invariants as

$$\boldsymbol{\mathcal{A}} = \sum_{i\in\mathcal{I}} W_i \frac{\partial^2 I_i}{\partial\mathbf{F}\partial\mathbf{F}} + \sum_{i,j\in\mathcal{I}} W_{ij} \frac{\partial I_i}{\partial\mathbf{F}} \otimes \frac{\partial I_j}{\partial\mathbf{F}}, \tag{4.1}$$

where $W_{ij} = \partial^2 W/\partial I_i \partial I_j$, $i, j \in \mathcal{I}$, and $\mathcal{I}$ is again the index set $\{1, 2, 3, 6, 7, 8, 9\}$.

The detailed (lengthy) expressions for the components of $\boldsymbol{\mathcal{A}}_0$ were given by Shams *et al.* (2011) for a general deformed configuration based on expressions for the second derivatives of the invariants, which were given in Appendix A of the latter paper. Here, we require only their specialization to the (undeformed) reference configuration ($\mathcal{B} \to \mathcal{B}_\mathrm{r}$), which, following Shams *et al.* (2011), yields

$$\begin{aligned}\mathcal{A}_{0piqj} &= \alpha_1(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}) + \alpha_2\delta_{ip}\delta_{jq} + \delta_{ij}\tau_{pq} + \beta_1(\delta_{ij}\tau_{pq} + \delta_{pq}\tau_{ij} + \delta_{iq}\tau_{jp} + \delta_{jp}\tau_{iq}) \\ &\quad + \beta_2(\delta_{ip}\tau_{jq} + \delta_{jq}\tau_{ip}) + \beta_3\tau_{ip}\tau_{jq} + \gamma_1(\delta_{ij}\tau_{pk}\tau_{kq} + \delta_{pq}\tau_{ik}\tau_{kj} + \delta_{iq}\tau_{jk}\tau_{kp} + \delta_{jp}\tau_{ik}\tau_{kq}) \\ &\quad + \gamma_2(\delta_{ip}\tau_{jk}\tau_{kq} + \delta_{jq}\tau_{ik}\tau_{kp}) + \gamma_3(\tau_{ip}\tau_{jk}\tau_{kq} + \tau_{jq}\tau_{ik}\tau_{kp}) + \gamma_4\tau_{ik}\tau_{kp}\tau_{jl}\tau_{lq},\end{aligned} \tag{4.2}$$

where the αs, βs, and γ s are defined by

$$\begin{aligned}\alpha_1 &= 2(W_1 + W_2), \quad \alpha_2 = 4(W_{11} + 4W_{12} + 4W_{22} + 2W_{13} + 4W_{23} + W_{33} - W_1 - W_2), \\ \beta_1 &= 2W_7, \quad \beta_2 = 4(W_{16} + 2W_{17} + 2W_{26} + 4W_{27} + W_{36} + 2W_{37}), \quad \gamma_1 = 2W_9, \\ \beta_3 &= 4(W_{66} + 4W_{67} + 4W_{77}), \quad \gamma_2 = 4(W_{18} + 2W_{19} + 2W_{28} + 4W_{29} + W_{38} + 2W_{39}), \\ \gamma_3 &= 4(W_{68} + 2W_{69} + 2W_{78} + 4W_{79}), \quad \gamma_4 = 4(W_{88} + 4W_{89} + 4W_{99}),\end{aligned} \tag{4.3}$$

all derivatives W_i and W_{ij} being evaluated in the reference configuration and use having been made of the connections (2.16). Note that in general, the expressions (4.3) may depend on the invariants $\operatorname{tr}\boldsymbol{\tau}$, $\operatorname{tr}(\boldsymbol{\tau}^2)$ and $\operatorname{tr}(\boldsymbol{\tau}^3)$. There are nine ($\boldsymbol{\tau}$-dependent) material parameters in the above, just as there are nine constants for an orthotropic linearly elastic material (see, for example, Ting, 1996), but additionally here the components τ_{ij} of $\boldsymbol{\tau}$ are present separately.

Note that when referred to axes that coincide with the principal axes of $\boldsymbol{\tau}$, the only non-zero components of (4.2) are given by

$$\begin{aligned}\mathcal{A}_{0iiii} &= 2\alpha_1 + \alpha_2 + (1 + 4\beta_1 + 2\beta_2)\tau_i + \beta_3\tau_i^2 + 2(2\gamma_1 + \gamma_2)\tau_i^2 + 2\gamma_3\tau_i^3 + \gamma_4\tau_i^4, \\ \mathcal{A}_{0iijj} &= \alpha_2 + \beta_2(\tau_i + \tau_j) + \beta_3\tau_i\tau_j + \gamma_2(\tau_i^2 + \tau_j^2) + \gamma_3(\tau_i + \tau_j)\tau_i\tau_j + \gamma_4\tau_i^2\tau_j^2, \\ \mathcal{A}_{0ijij} &= \alpha_1 + \tau_i + \beta_1(\tau_i + \tau_j) + \gamma_1(\tau_i^2 + \tau_j^2) = \mathcal{A}_{0ijji} + \tau_i,\end{aligned} \tag{4.4}$$

where there are no sums on repeated indices, $i \neq j$, and τ_i, $i = 1, 2, 3$, are the principal values of $\boldsymbol{\tau}$ (in general, there are 15 non-zero components of $\boldsymbol{\mathcal{A}}_0$ in total, dependent on nine material parameters and three principal stresses). In the linear specialization of the above only the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ are retained along with τ_1, τ_2, τ_3, although we should strictly expand α_1 and α_2 as linear functions of $\boldsymbol{\tau}$. Then there are nine constants involved, specifically $\alpha_1(\mathbf{0})$, $\alpha_2(\mathbf{0})$, $\beta_1(\mathbf{0})$, $\beta_2(\mathbf{0})$, $\alpha_{1,i}(\mathbf{0})$, $\alpha_{2,i}(\mathbf{0})$, $i = 1, 2, 3$, and the three principal initial stresses τ_1, τ_2, τ_3, where $\mathbf{0} = (0, 0, 0)$ is the value of (τ_1, τ_2, τ_3) for zero initial stress and $_{,i}$ signifies differentiation with respect to τ_i, $i = 1, 2, 3$. Note that since the coefficients in (4.4) are symmetric functions of (τ_1, τ_2, τ_3), the constants $\alpha_{1,i}(\mathbf{0})$ and $\alpha_{2,i}(\mathbf{0})$ are independent of i.

We now specialize the above to consider three specific forms of $\boldsymbol{\tau}$, corresponding to isotropic initial stress, uniaxial initial stress and planar shear initial stress.

4.1 *Isotropic initial stress*

Suppose that $\boldsymbol{\tau}$ is isotropic and write $\boldsymbol{\tau} = \tau\mathbf{I}$, where $\mathbf{I}$ is again the identity tensor and $\tau > 0\,(< 0)$ corresponds to hydrostatic tension (pressure). Then (4.2) reduces to the compact form

$$\mathcal{A}_{0piqj} = \tau\delta_{ij}\delta_{pq} + \alpha(\tau)(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{pj}) + \beta(\tau)\delta_{pi}\delta_{qj}, \tag{4.5}$$

where the notations $\alpha(\tau)$ and $\beta(\tau)$ are defined by

$$\alpha(\tau) = \alpha_1(\tau) + 2\tau\beta_1(\tau) + 2\tau^2\gamma_1(\tau), \tag{4.6}$$

$$\beta(\tau) = \alpha_2(\tau) + 2\tau\beta_2(\tau) + \tau^2[\beta_3(\tau) + 2\gamma_2(\tau)] + 2\tau^3\gamma_3(\tau) + \tau^4\gamma_4(\tau), \tag{4.7}$$

and we note that, by virtue of the specializations $\operatorname{tr}\boldsymbol{\tau} = 3\tau, \operatorname{tr}(\boldsymbol{\tau}^2) = 3\tau^2, \operatorname{tr}(\boldsymbol{\tau}^3) = 3\tau^3$, the coefficients $\alpha_1, \ldots, \gamma_4$ are now (in general) functions of the single parameter τ, which is indicated above by inclusion of the argument τ.

4.2 *Uniaxial initial stress*

Here, we take $\boldsymbol{\tau} = \tau\mathbf{a} \otimes \mathbf{a}$, where $\mathbf{a}$ is a fixed unit vector (the direction of the uniaxial stress, which is a tension for $\tau > 0$ and a compressive stress for $\tau < 0$). In this case, the components of $\boldsymbol{\mathcal{A}}_0$ may be expressed in the form

$$\mathcal{A}_{0piqj} = \alpha_1(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{pj}) + \alpha_2\delta_{pi}\delta_{qj} + \tau(\beta_1 + \tau\gamma_1)(\delta_{ij}a_pa_q + \delta_{pq}a_ia_j + \delta_{iq}a_pa_j + \delta_{pj}a_ia_q) + \tau\delta_{ij}a_pa_q + \tau(\beta_2 + \tau\gamma_2)(\delta_{pi}a_qa_j + \delta_{qj}a_pa_i) + \tau^2(\beta_3 + 2\tau\gamma_3 + \tau^2\gamma_4)a_pa_ia_qa_j. \tag{4.8}$$

Again the coefficients $\alpha_1, \ldots, \gamma_4$ depend on τ in general but for the sake of brevity, this has been left implicit here. Note that, in addition to τ, there are five separate (combinations of) parameters, namely $\alpha_1, \alpha_2, \beta_1 + \tau\gamma_1, \beta_2 + \tau\gamma_2, \beta_3 + \tau\gamma_3 + \tau^2\gamma_4$ and we recall that in classical transversely isotropic linear elasticity, there are five material constants (see, for example, Ting, 1996).

Without loss of generality, we may take $\mathbf{a}$ to coincide with the axis $\mathbf{e}_1$. Then the components of $\boldsymbol{\mathcal{A}}_0$ are listed as

$$\mathcal{A}_{01111} = 2\alpha_1 + \alpha_2 + (1 + 4\beta_1 + 2\beta_2)\tau + (4\gamma_1 + 2\gamma_2 + \beta_3)\tau^2 + 2\gamma_3\tau^3 + \gamma_4\tau^4, \tag{4.9}$$

$$\mathcal{A}_{0iiii} = 2\alpha_1 + \alpha_2, \quad \mathcal{A}_{0iijj} = \alpha_2, \quad \mathcal{A}_{011ii} = \alpha_2 + \beta_2\tau + \gamma_2\tau^2, \quad i, j \in \{2, 3\}, i \neq j, \tag{4.10}$$

$$\mathcal{A}_{0i1i1} = \mathcal{A}_{01ii1} = \mathcal{A}_{0i11i} = \alpha_1 + \beta_1\tau + \gamma_1\tau^2, \quad i \in \{2,3\}, \tag{4.11}$$

$$\mathcal{A}_{01i1i} = \alpha_1 + (1+\beta_1)\tau + \gamma_1\tau^2, \quad \mathcal{A}_{0ijij} = \mathcal{A}_{0ijji} = \alpha_1, \quad i,j \in \{2,3\}, i \neq j, \tag{4.12}$$

for later reference. Note that in the linear specialization, there remain only seven independent constants, namely $\alpha_1(0)$, $\alpha_2(0)$, $\beta_1(0)$, $\beta_2(0)$, $\alpha_1'(0)$, $\alpha_2'(0)$, with argument $\tau = 0$, and τ, where the prime indicates differentiation with respect to τ.

4.3 *Planar shear initial stress*

Consider planar shear stress in the (x_1, x_2) plane of the form $\boldsymbol{\tau} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$. Then the only non-zero components of $\boldsymbol{\mathcal{A}}_0$ are written as

$$\mathcal{A}_{01111} = \mathcal{A}_{02222} = 2\alpha_1 + \alpha_2 + 2(2\gamma_1 + \gamma_2)\tau^2 + \gamma_4\tau^4, \quad \mathcal{A}_{03333} = 2\alpha_1 + \alpha_2, \tag{4.13}$$

$$\mathcal{A}_{01122} = \alpha_2 + 2\gamma_2\tau^2 + \gamma_4\tau^4, \quad \mathcal{A}_{01133} = \mathcal{A}_{02233} = \alpha_2, \tag{4.14}$$

$$\mathcal{A}_{0ijij} = \mathcal{A}_{0ijji} = \alpha_1 + (\beta_3 + 2\gamma_1)\tau^2, \quad i,j \in \{1,2\}, i \neq j, \tag{4.15}$$

$$\mathcal{A}_{0i3i3} = \mathcal{A}_{03i3i} = \mathcal{A}_{03ii3} = \alpha_1 + \gamma_1\tau^2, \quad i \in \{1,2\}, \tag{4.16}$$

$$\mathcal{A}_{0iiij} = (2\beta_1 + \beta_2)\tau + \gamma_3\tau^3, \quad \mathcal{A}_{0jiii} = \mathcal{A}_{0iiij} + \tau, \quad i,j \in \{1,2\}, i \neq j. \tag{4.17}$$

In the linear specialization, there are now five constants, $\alpha_1(0)$, $\alpha_2(0)$, $2\beta_1(0) + \beta_2(0)$, $\alpha_1'(0)$, $\alpha_2'(0)$, in addition to τ.

5. Dependence of elastic moduli on initial stress

5.1 *Isotropic initial stress*

In the connection (3.4), we now specialize the Cauchy stress $\boldsymbol{\sigma}$ to the initial stress $\boldsymbol{\tau}$, so that

$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{S}}_0 + \mathbf{L}\boldsymbol{\tau} - (\mathrm{tr}\,\mathbf{L})\boldsymbol{\tau}. \tag{5.1}$$

It follows on use of (4.5) that, for an isotropic initial stress $\boldsymbol{\tau} = \tau\mathbf{I}$,

$$\dot{\mathbf{S}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} = \tau\mathbf{L}^\top + \alpha(\tau)(\mathbf{L} + \mathbf{L}^\top) + \beta(\tau)(\mathrm{tr}\,\mathbf{L})\mathbf{I} \tag{5.2}$$

and hence that

$$\dot{\boldsymbol{\sigma}} = [\beta(\tau) - \tau](\mathrm{tr}\,\mathbf{L})\mathbf{I} + [\alpha(\tau) + \tau](\mathbf{L} + \mathbf{L}^\top). \tag{5.3}$$

Since the initial stress is purely isotropic, we can therefore identify the *stress-dependent Lamé moduli*, which we denote as $\lambda(\tau)$ and $\mu(\tau)$, as

$$\lambda(\tau) = \beta(\tau) - \tau = \alpha_2(\tau) + 2\tau\beta_2(\tau) + \tau^2[\beta_3(\tau) + 2\gamma_2(\tau)] + 2\tau^3\gamma_3(\tau) + \tau^4\gamma_4(\tau) - \tau, \tag{5.4}$$

$$\mu(\tau) = \alpha(\tau) + \tau = \alpha_1(\tau) + 2\tau\beta_1(\tau) + 2\tau^2\gamma_1(\tau) + \tau. \tag{5.5}$$

For incremental simple shear, we may, without loss of generality, restrict attention to the (x_1, x_2) plane. If the shear is in the x_1 direction with amount of shear $L_{ij} = L_{12}\delta_{1i}\delta_{2j}$, then the corresponding incremental nominal and Cauchy stress components are equal and given by $\dot{S}_{021} = \dot{\sigma}_{12} = \mathcal{A}_{02121}$

$L_{12} = \mu(\tau)L_{12}$. Similarly, for incremental simple shear in the x_2 direction with $L_{ij} = L_{21}\delta_{2i}\delta_{1j}$, we have $\dot{S}_{012} = \dot{\sigma}_{12} = \mathcal{A}_{01212}L_{21} = \mu(\tau)L_{21}$, and from (4.5) and (5.5), $\mathcal{A}_{02121} = \mathcal{A}_{01212} = \mu(\tau)$.

For incremental pure dilatation with $L_{11} = L_{22} = L_{33} = (\mathrm{tr}\,\mathbf{L})/3$, we obtain

$$\mathrm{tr}\,\dot{\boldsymbol{\sigma}} = [3\lambda(\tau) + 2\mu(\tau)]\mathrm{tr}\,\mathbf{L}, \tag{5.6}$$

and this enables us to define the *stress-dependent bulk modulus* $\kappa(\tau)$, analogously to the classical formula, as

$$\kappa(\tau) = \lambda(\tau) + \frac{2}{3}\mu(\tau). \tag{5.7}$$

Note that from (5.1), we obtain $\mathrm{tr}\,\dot{\mathbf{S}}_0 = \mathrm{tr}\,\dot{\boldsymbol{\sigma}} + 2\tau\,\mathrm{tr}\,\mathbf{L}$, but use of $\mathrm{tr}\,\dot{\mathbf{S}}_0$ instead of $\mathrm{tr}\,\dot{\boldsymbol{\sigma}}$ does not give the correct form of the bulk modulus. Also, by using (3.5) with $\boldsymbol{\sigma} = \boldsymbol{\tau} = \tau\mathbf{I}$ and (5.2), we obtain

$$\dot{\mathbf{T}}_0 = \alpha(\tau)(\mathbf{L} + \mathbf{L}^\top) + \beta(\tau)(\mathrm{tr}\,\mathbf{L})\mathbf{I}. \tag{5.8}$$

Thus, it is clear that because of the dependence on the initial stress, different choices of stress measure lead to different possible definitions of the stress-dependent elastic moduli.

The correct definitions for the stress-dependent Lamé moduli are (5.4) and (5.5). If the second Piola–Kirchhoff stress is used instead, then $\mu(\tau)$ and $\lambda(\tau)$ would be replaced by $\alpha(\tau)$ and $\beta(\tau)$, respectively. This was effectively what was done in the paper by Tang (1967), although he worked in terms of Young's modulus and Poisson's ratio. This identification of the Lamé moduli leads to erroneous results for the speeds of homogeneous plane waves, as we shall show in Section 6.

If there is no initial stress (reduction to the classical case), we denote the classical moduli by $\lambda_0, \mu_0, \kappa_0$, so that $\lambda_0 = \lambda(0) = \alpha_2(0)$, $\mu_0 = \mu(0) = \alpha_1(0)$ and $\kappa_0 = \kappa(0)$. If the initial stress is small in magnitude, then we may linearize the expressions (5.4), (5.5) and (5.7) to obtain

$$\lambda(\tau) \simeq \lambda_0 + [\alpha_2'(0) + 2\beta_2(0) - 1]\tau, \tag{5.9}$$

$$\mu(\tau) \simeq \mu_0 + [\alpha_1'(0) + 2\beta_1(0) + 1]\tau, \tag{5.10}$$

$$\kappa(\tau) \simeq \kappa_0 + [2\alpha_1'(0) + 3\alpha_2'(0) + 4\beta_1(0) + 6\beta_2(0) - 1]\tau/3. \tag{5.11}$$

5.2 *Uniaxial initial stress*

When the initial stress is uniaxial, the subsequent incremental response is transversely isotropic in nature. Then it is appropriate to work in terms of Young's moduli and Poisson's ratios. In order to determine these, we need to examine both triaxial incremental deformations without shear and separate incremental shear deformations. Consider first the normal components of $\mathbf{L}$, written L_{11}, L_{22}, L_{33}, and take $\mathbf{e}_1$ to be the direction of the uniaxial initial stress. Then, bearing in mind the symmetry in (4.10), the corresponding incremental nominal stresses are

$$\dot{S}_{011} = \mathcal{A}_{01111}L_{11} + \mathcal{A}_{01122}(L_{22} + L_{33}), \tag{5.12}$$

$$\dot{S}_{022} = \mathcal{A}_{01122}L_{11} + \mathcal{A}_{02222}L_{22} + \mathcal{A}_{02233}L_{33}, \tag{5.13}$$

$$\dot{S}_{033} = \mathcal{A}_{01122}L_{11} + \mathcal{A}_{02233}L_{22} + \mathcal{A}_{02222}L_{33}. \tag{5.14}$$

The (incremental) Poisson's ratio ν_{12} (and hence ν_{13} by symmetry) is obtained by setting $\dot{S}_{022} = \dot{S}_{033} = 0$ and using the resulting symmetry $L_{33} = L_{22}$ and $L_{22} = -\nu_{12}L_{11}$ to obtain

$$\nu_{12} = \mathcal{A}_{01122}/(\mathcal{A}_{02222} + \mathcal{A}_{02233}). \tag{5.15}$$

Then,

$$\dot{S}_{011} = (\mathcal{A}_{01111} - 2\nu_{12}\mathcal{A}_{01122})L_{11} \tag{5.16}$$

and the (incremental) Young's modulus E_1 can be read off as

$$E_1 = \mathcal{A}_{01111} - 2\nu_{12}\mathcal{A}_{01122} = \mathcal{A}_{01111} - 2\mathcal{A}_{01122}^2/(\mathcal{A}_{02222} + \mathcal{A}_{02233}). \tag{5.17}$$

To obtain $\nu_{21} = \nu_{31}$, $\nu_{23} = \nu_{32}$ and $E_2 = E_3$, on the other hand, we set $\dot{S}_{011} = \dot{S}_{033} = 0$, with $L_{11} = -\nu_{21}L_{22}$, $L_{33} = -\nu_{23}L_{22}$, and hence

$$\nu_{21}\mathcal{A}_{01111} + \nu_{23}\mathcal{A}_{01122} = \mathcal{A}_{01122}, \quad \nu_{21}\mathcal{A}_{01122} + \nu_{23}\mathcal{A}_{02222} = \mathcal{A}_{02233}, \tag{5.18}$$

and

$$\dot{S}_{022} = (\mathcal{A}_{02222} - \nu_{21}\mathcal{A}_{01122} - \nu_{23}\mathcal{A}_{02233})L_{22} = E_2 L_{22}. \tag{5.19}$$

These yield

$$\nu_{21} = \frac{\mathcal{A}_{01122}(\mathcal{A}_{02222} - \mathcal{A}_{02233})}{\mathcal{A}_{01111}\mathcal{A}_{02222} - \mathcal{A}_{01122}^2}, \quad \nu_{23} = \frac{\mathcal{A}_{01111}\mathcal{A}_{02233} - \mathcal{A}_{01122}^2}{\mathcal{A}_{01111}\mathcal{A}_{02222} - \mathcal{A}_{01122}^2} \tag{5.20}$$

and

$$E_2 = (\mathcal{A}_{02222} - \mathcal{A}_{02233})\frac{\mathcal{A}_{01111}(\mathcal{A}_{02222} + \mathcal{A}_{02233}) - 2\mathcal{A}_{01122}^2}{\mathcal{A}_{01111}\mathcal{A}_{02222} - \mathcal{A}_{01122}^2}. \tag{5.21}$$

The connection

$$E_2/\nu_{21} = E_1/\nu_{12} \tag{5.22}$$

then follows, as in the classical linear theory. We note, however, that if the increments of the Cauchy stress were used in the definitions of the stress-dependent Poisson's ratios and Young's moduli instead of the nominal stress (which is entirely legitimate), then this would not follow. There are now four independent material parameters: $\nu_{12} = \nu_{13}$, $\nu_{21} = \nu_{31}$, $\nu_{23} = \nu_{32}$ and E_1, for example, with $E_2 = E_3$ given by (5.22).

For the incremental shear response in the plane of symmetry, we have

$$\dot{\sigma}_{23} = \dot{S}_{023} = \mathcal{A}_{02323}L_{32} + \mathcal{A}_{02332}L_{23} = \alpha_1(L_{23} + L_{32}), \tag{5.23}$$

according to $(4.12)_2$, and hence α_1 is the shear modulus in the plane of symmetry. In fact, it is straightforward to show that it can be expressed in terms of the other parameters as

$$\alpha_1 = E_2/2(1 + \nu_{23}), \tag{5.24}$$

similarly to the situation in the classical theory.

However, when it comes to shear in a plane that contains the axis $\mathbf{e}_1$ the situation differs from the classical one because of the influence of τ. We have

$$\dot{\sigma}_{12} = \dot{S}_{012} = \dot{S}_{021} + \tau L_{21} = [\alpha_1 + \tau(\beta_1 + \tau\gamma_1) + \tau]L_{21} + [\alpha_1 + \tau(\beta_1 + \tau\gamma_1)]L_{12}. \tag{5.25}$$

For shear in the x_1 direction with incremental simple shear $L_{ij} = L_{12}\delta_{1i}\delta_{2j}$, we obtain

$$\dot{\sigma}_{12} = \dot{S}_{021} = [\alpha_1 + \tau(\beta_1 + \tau\gamma_1)]L_{12}, \tag{5.26}$$

while for shear transverse to the x_1 direction with incremental simple shear $L_{ij} = L_{21}\delta_{2i}\delta_{1j}$, we have

$$\dot{\sigma}_{12} = \dot{S}_{012} = [\alpha_1 + \tau(\beta_1 + \tau\gamma_1) + \tau]L_{21}, \tag{5.27}$$

and the associated shear moduli are $\alpha_1 + \tau(\beta_1 + \tau\gamma_1)$ and $\alpha_1 + \tau(\beta_1 + \tau\gamma_1) + \tau$, respectively.

Thus, in total, there are six coefficients that are functions of τ, but when linearized in τ there remain six constants, namely $\alpha_1(0)$, $\alpha_2(0)$, $\beta_1(0)$, $\beta_2(0)$, $\alpha_1'(0)$ and $\alpha_2'(0)$, together with τ.

5.3 *Planar shear initial stress*

If the initial stress lies in the (x_1, x_2) plane and is a pure shear stress of amount τ, then $\boldsymbol{\tau} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$ and the principal values of $\boldsymbol{\tau}$ are $\pm\tau$. The principal axes of $\boldsymbol{\tau}$ bisect the background axes $\mathbf{e}_1$ and $\mathbf{e}_2$, along $\hat{\mathbf{e}}_{1,2} = (\mathbf{e}_1 \pm \mathbf{e}_2)/\sqrt{2}$, say. Here, we therefore take as our axes of reference the principal axes $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3 = \mathbf{e}_3$; the associated components of $\boldsymbol{\mathcal{A}}_0$ are

$$\hat{\mathcal{A}}_{01111} = 2\alpha_1 + \alpha_2 + (1 + 4\beta_1 + 2\beta_2)\tau + (\beta_3 + 4\gamma_1 + 2\gamma_2)\tau^2 + 2\gamma_3\tau^3 + \gamma_4\tau^4, \tag{5.28}$$
$$\hat{\mathcal{A}}_{02222} = 2\alpha_1 + \alpha_2 - (1 + 4\beta_1 + 2\beta_2)\tau + (\beta_3 + 4\gamma_1 + 2\gamma_2)\tau^2 - 2\gamma_3\tau^3 + \gamma_4\tau^4, \tag{5.29}$$
$$\hat{\mathcal{A}}_{03333} = 2\alpha_1 + \alpha_2, \quad \hat{\mathcal{A}}_{01122} = \alpha_2 - \beta_3\tau^2 + 2\gamma_2\tau^2 + \gamma_4\tau^4, \tag{5.30}$$
$$\hat{\mathcal{A}}_{01133} = \alpha_2 + \beta_2\tau + \gamma_2\tau^2, \quad \hat{\mathcal{A}}_{02233} = \alpha_2 - \beta_2\tau + \gamma_2\tau^2, \tag{5.31}$$
$$\hat{\mathcal{A}}_{01221} = \alpha_1 + 2\gamma_1\tau^2, \quad \hat{\mathcal{A}}_{01212} = \alpha_1 + \tau + 2\gamma_1\tau^2, \quad \hat{\mathcal{A}}_{02121} = \alpha_1 - \tau + 2\gamma_1\tau^2, \tag{5.32}$$
$$\hat{\mathcal{A}}_{03131} = \hat{\mathcal{A}}_{03113} = \alpha_1 + \tau\beta_1 + \tau^2\gamma_1, \quad \hat{\mathcal{A}}_{01313} = \alpha_1 + \tau + \tau\beta_1 + \tau^2\gamma_1, \tag{5.33}$$
$$\hat{\mathcal{A}}_{03232} = \hat{\mathcal{A}}_{03223} = \alpha_1 - \tau\beta_1 + \tau^2\gamma_1, \quad \hat{\mathcal{A}}_{02323} = \alpha_1 - \tau - \tau\beta_1 + \tau^2\gamma_1. \tag{5.34}$$

Components referred to principal axes are indicated by a superposed hat. We also have $\mathrm{tr}\,\boldsymbol{\tau} = \mathrm{tr}\,(\boldsymbol{\tau}^3) = 0$, $\mathrm{tr}\,(\boldsymbol{\tau}^2) = 2\tau^2$, and, in general, all the coefficients $\alpha_1, \dots, \gamma_4$ depend on τ.

To illustrate the results in this case, we consider the restriction to incremental plane strain with $\hat{L}_{3i} = \hat{L}_{i3} = 0$, $i = 1, 2, 3$. The components of the incremental nominal stress are then given by

$$\hat{\dot{S}}_{011} = \hat{\mathcal{A}}_{01111}\hat{L}_{11} + \hat{\mathcal{A}}_{01122}\hat{L}_{22}, \quad \hat{\dot{S}}_{022} = \hat{\mathcal{A}}_{01122}\hat{L}_{11} + \hat{\mathcal{A}}_{02222}\hat{L}_{22} \tag{5.35}$$

for biaxial deformation parallel to the principal axes, and

$$\hat{\dot{S}}_{012} = (\alpha_1 + 2\gamma_1\tau^2)(\hat{L}_{12} + \hat{L}_{21}) + \tau\hat{L}_{21}, \quad \hat{\dot{S}}_{021} = (\alpha_1 + 2\gamma_1\tau^2)(\hat{L}_{12} + \hat{L}_{21}) - \tau\hat{L}_{12} \tag{5.36}$$

for shearing deformations.

The plane strain Poisson's ratios and Young's modulus E_1 are then deduced as

$$\nu_{12} = \hat{\mathcal{A}}_{01122}/\hat{\mathcal{A}}_{02222}, \quad \nu_{21} = \hat{\mathcal{A}}_{01122}/\hat{\mathcal{A}}_{01111}, \quad E_1 = \hat{\mathcal{A}}_{01111} - \hat{\mathcal{A}}_{01122}^2/\hat{\mathcal{A}}_{02222} \tag{5.37}$$

and, as in (5.22), $E_2 = E_1\nu_{21}/\nu_{12}$. The shear moduli are $\alpha_1 - \tau + 2\gamma_1\tau^2$ and $\alpha_1 + \tau + 2\gamma_1\tau^2$ for shear in the $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ directions, respectively.

When $\tau = 0$, we recover the classical results for plane strain isotropy: Poisson's ratio is $\nu_{12} = \nu_{21} = \lambda_0/(\lambda_0 + 2\mu_0)$, and Young's modulus is $E_1 = E_2 = 4\mu_0(\lambda_0 + \mu_0)/(\lambda_0 + 2\mu_0)$.

6. The effect of initial stress on infinitesimal wave propagation

The theory of small amplitude (incremental) deformations or motions superimposed on a static finite deformation is well established but has received relatively little attention in the case of an initially stressed material with or without an accompanying finite deformation except for works based on Biot's theory in the context of linear elasticity. Here, we consider incremental motions in an infinite homogeneous medium subject to a homogeneous initial stress with $\mathcal{A}_{0piqj}$ having the form given in (4.2) and various specializations of that form.

From (3.12), we recall that the equation of incremental motion is

$$\mathcal{A}_{0piqj} u_{j,pq} = \rho u_{i,tt}. \tag{6.1}$$

Consider a homogeneous plane wave of the form

$$\mathbf{u} = \mathbf{m} f(\mathbf{n} \cdot \mathbf{x} - v t), \tag{6.2}$$

where $\mathbf{m}$ is a fixed unit vector (the polarization vector), f is a function of the argument $\mathbf{n} \cdot \mathbf{x} - vt$ with appropriate regularity, $\mathbf{n}$ is a unit vector in the direction of propagation and v is the wave speed. Substitution into the equation of motion (6.1) (after dropping f'', which is assumed to be non-zero) leads to

$$\mathcal{A}_{0piqj} n_p n_q m_j = \rho v^2 m_i. \tag{6.3}$$

The associated *acoustic tensor* $\mathbf{Q}(\mathbf{n})$ has components defined by

$$Q_{ij}(\mathbf{n}) = \mathcal{A}_{0piqj} n_p n_q \tag{6.4}$$

and enables the *propagation condition* (6.3) to be written compactly as

$$\mathbf{Q}(\mathbf{n})\mathbf{m} = \rho v^2 \mathbf{m}. \tag{6.5}$$

For any given direction of propagation $\mathbf{n}$, we have a 3D symmetric algebraic eigenvalue problem for determining ρv^2 and $\mathbf{m}$. Because of the symmetry, there are three mutually orthogonal eigenvectors $\mathbf{m}$ corresponding to the directions of displacement and the (three) values of ρv^2 are obtained from the characteristic equation

$$\det[\mathbf{Q}(\mathbf{n}) - \rho v^2 \mathbf{I}] = 0. \tag{6.6}$$

If $\mathbf{m}$ is known, then ρv^2 is given by

$$\rho v^2 = [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} \tag{6.7}$$

and corresponds to a real wave speed provided $\rho v^2 > 0$, which is guaranteed if the *strong ellipticity condition* holds, i.e. if

$$[\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} \equiv \mathcal{A}_{0piqj} n_p n_q m_i m_j > 0 \quad \text{for all non-zero vectors } \mathbf{m}, \mathbf{n}. \tag{6.8}$$

Then, a triad of waves with mutually orthogonal polarizations can propagate for any direction of propagation $\mathbf{n}$. Henceforth, we assume that the strong ellipticity condition holds. For detailed discussion of strong ellipticity, we refer to Truesdell & Noll (1965) and Ogden (1984), for example.

In the following, we examine the effect of initial stress on the propagation of plane waves for the three examples of initial stress considered in Sections 4 and 5, and for this purpose, we give explicit expressions for $\mathbf{Q}(\mathbf{n})$ in each case.

6.1 *Isotropic initial stress*

For the form of $\boldsymbol{\mathcal{A}}_0$ given by (4.5), with the connections (5.4) and (5.5) and (6.4), we obtain simply

$$\mathbf{Q}(\mathbf{n}) = \mu(\tau)\mathbf{I} + [\lambda(\tau) + \mu(\tau)]\mathbf{n} \otimes \mathbf{n}. \tag{6.9}$$

As in the classical theory of isotropic elasticity with no initial stress, there exists a longitudinal wave with speed v_{L}, say, and two transverse waves with equal speeds v_{T}, say, for any direction of propagation. With dependence on τ these are given by

$$\rho v_{\mathrm{L}}^2 = \lambda(\tau) + 2\mu(\tau), \quad \rho v_{\mathrm{T}}^2 = \mu(\tau). \tag{6.10}$$

For sufficiently small initial stress, we may linearize these expressions to give

$$\lambda(\tau) + 2\mu(\tau) \simeq \lambda_0 + 2\mu_0 + [2\alpha_1'(0) + \alpha_2'(0) + 4\beta_1(0) + 2\beta_2(0) + 1]\tau, \tag{6.11}$$

$$\mu(\tau) \simeq \mu_0 + [\alpha_1'(0) + 2\beta_1(0) + 1]\tau, \tag{6.12}$$

where $\alpha_1(0) = \mu_0$, $\alpha_2(0) = \lambda_0$.

Note that if, as in Tang (1967), the isotropic moduli were defined based on the increment in the second Piola–Kirchhoff stress according to (5.8), then $\mu(\tau)$ and $\lambda(\tau)+2\mu(\tau)$ would have to be replaced by $\mu(\tau)+\tau$ and $\lambda(\tau)+2\mu(\tau)+\tau$, respectively, leading to the erroneous conclusion of Tang (1967) that when the elastic moduli are independent of the initial stress, the wave speeds are given by $\rho v_{\mathrm{T}}^2 = \mu_0 - P$, $\rho v_{\mathrm{L}}^2 = \lambda_0 + 2\mu_0 - P$ for the case of a hydrostatic pressure ($\tau = -P$). Results such as these are not supported by the data shown in Tables 1 and 2.

For a general isotropic elastic material under hydrostatic pressure Truesdell (1961) obtained expressions for the speeds of longitudinal and transverse waves (see also Truesdell & Noll, 1965, Section 75).

6.2 *Uniaxial initial stress*

For uniaxial initial stress, the acoustic tensor is given by

$$\mathbf{Q}(\mathbf{n}) = A\mathbf{I} + B\mathbf{a} \otimes \mathbf{a} + C\mathbf{n} \otimes \mathbf{n} + D(\mathbf{n} \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{n}), \tag{6.13}$$

where

$$\begin{aligned} A &= \alpha_1 + [1 + \beta_1 + \gamma_1\tau]\tau(\mathbf{n}\cdot\mathbf{a})^2, \\ B &= \beta_1\tau + \gamma_1\tau^2 + [\beta_3 + 2\gamma_3\tau + \gamma_4\tau^2]\tau^2(\mathbf{n}\cdot\mathbf{a})^2, \\ C &= \alpha_1 + \alpha_2, \\ D &= [\beta_1 + \beta_2 + (\gamma_1 + \gamma_2)\tau]\tau(\mathbf{n}\cdot\mathbf{a}). \end{aligned} \tag{6.14}$$

If $\mathbf{n} = \mathbf{a}$ (propagation along the direction of uniaxial stress), then there exists a longitudinal wave with speed v_{11} given by

$$\rho v_{11}^2 = A + B + C + 2D = 2\alpha_1 + \alpha_2 + (1 + 4\beta_1 + 2\beta_2)\tau + (\beta_3 + 4\gamma_1 + 2\gamma_2)\tau^2 + 2\gamma_3\tau^3 + \gamma_4\tau^4, \tag{6.15}$$

and two transverse waves with equal speeds v_{12} given by

$$\rho v_{12}^2 = A = \alpha_1 + (1 + \beta_1)\tau + \gamma_1\tau^2. \tag{6.16}$$

These formulas are consistent with the formulas (5.20) and (5.23) in Shams *et al.* (2011) relating to propagation along a principal axis of $\boldsymbol{\tau}$ except that in (5.20) there is a misprint (the coefficient of γ_1 should be 4 instead of 3—this arises from the fact that in the expression (5.18) for a in the latter paper γ_1 should be $2\gamma_1$). The linearized forms of (6.15) and (6.16) are

$$\rho v_{11}^2 = \lambda_0 + 2\mu_0 + [2\alpha_1'(0) + \alpha_2'(0) + 4\beta_1(0) + 2\beta_2(0) + 1]\tau, \tag{6.17}$$

$$\rho v_{12}^2 = \mu_0 + [\alpha_1'(0) + \beta_1(0) + 1]\tau, \tag{6.18}$$

respectively.

On the other hand, if $\mathbf{n} \cdot \mathbf{a} = 0$ (propagation transverse to the direction of uniaxial stress), then

$$A = \alpha_1, \quad B = \beta_1\tau + \gamma_1\tau^2, \quad C = \alpha_1 + \alpha_2, \quad D = 0, \tag{6.19}$$

and a longitudinal wave exists with speed v_{22} given by $\rho v_{22}^2 = A + C$. There are also two transverse waves, with polarizations along and perpendicular to $\mathbf{a}$ and speeds v_{21} and v_{23} given by

$$\rho v_{21}^2 = A + B = \alpha_1 + \beta_1\tau + \gamma_1\tau^2, \quad \rho v_{23}^2 = A = \alpha_1, \tag{6.20}$$

respectively. We notice immediately that

$$\rho v_{12}^2 - \rho v_{21}^2 = \tau \tag{6.21}$$

exactly, as expected from the property (3.8). This well-known relationship (see, for example, Biot, 1965; Man & Lu, 1987; Hoger, 1993a; Norris, 1998) forms the basis of an experimental acoustic identification of a solid with anisotropy due to an initial stress, as opposed to a general anisotropic linearly elastic solid without initial stress, for which $\rho v_{12}^2 = \rho v_{21}^2 = c_{66}$ (Voigt notation). We note that, even earlier, Biot (1940a) obtained separate expressions for, in the present notation, v_{12} and v_{21} from which the above relationship may also be deduced. Finally, we note that the linearized forms of ρv_{22}^2 and (6.20) are

$$\rho v_{22}^2 = \lambda_0 + 2\mu_0 + [2\alpha_1'(0) + \alpha_2'(0)]\tau, \quad \rho v_{21}^2 = \mu_0 + [\alpha_1'(0) + \beta_1(0)]\tau, \quad \rho v_{23}^2 = \mu_0 + \alpha_1'(0)\tau. \tag{6.22}$$

If the propagation takes place neither along the direction of uniaxial stress nor perpendicular to it, then several possibilities arise, which are detailed in Appendix A.

6.3 *Planar shear initial stress*

Here, we consider wave propagation in the plane of shear—the (x_1, x_2) plane. The relevant components of the acoustic tensor are then given by

$$\begin{aligned}
Q_{11} &= \alpha_1 + [\alpha_1 + \alpha_2 + 2(2\gamma_1 + \gamma_2)\tau^2 + \gamma_4\tau^4]n_1^2 + 2(1 + 2\beta_1 + \beta_2 + \gamma_3\tau^2)\tau n_1 n_2 + (\beta_3 + 2\gamma_1)\tau^2 n_2^2, \\
Q_{22} &= \alpha_1 + [\alpha_1 + \alpha_2 + 2(2\gamma_1 + \gamma_2)\tau^2 + \gamma_4\tau^4]n_2^2 + 2(1 + 2\beta_1 + \beta_2 + \gamma_3\tau^2)\tau n_1 n_2 + (\beta_3 + 2\gamma_1)\tau^2 n_1^2, \\
Q_{12} &= (2\beta_1 + \beta_2)\tau + \gamma_3\tau^3 + [\alpha_1 + \alpha_2 + (2\gamma_1 + 2\gamma_2 + \beta_3)\tau^2 + \gamma_4\tau^4]n_1 n_2,
\end{aligned}$$

referred to background axes (not the principal axes considered in Section 5.3). As already noted, we assume that the strong ellipticity condition holds so the wave speeds are real. As in the previous section, we work in the (x_1, x_2) plane with $n_3 = m_3 = 0$ and set $n_1 = \cos\theta, n_2 = \sin\theta$ and $m_1 = \cos\phi$,

$m_2 = \sin\phi$. Then, by eliminating the wave speed from the propagation condition, we may apply (A.8) from Appendix A, recast as

$$\tan 2\phi = \frac{2(2\beta_1 + \beta_2 + \gamma_3\tau^2)\tau + [\alpha_1 + \alpha_2 + (2\gamma_1 + 2\gamma_2 + \beta_3)\tau^2 + \gamma_4\tau^4]\sin 2\theta}{[\alpha_1 + \alpha_2 + (2\gamma_1 + 2\gamma_2 - \beta_3 + \gamma_4\tau^2)\tau^2]\cos 2\theta}, \tag{6.23}$$

which gives ϕ for any given θ.

The (in-plane) wave speeds are given by

$$\rho v^2 = \frac{1}{2}\left[Q_{11} + Q_{22} \pm \sqrt{(Q_{11} - Q_{22})^2 + 4Q_{12}^2}\right]. \tag{6.24}$$

For definiteness, it is interesting to consider the situation in which τ is small and (6.23) is linearized in τ, which leads to

$$(\alpha_1 + \alpha_2)\sin(2\phi - 2\theta) = 2(2\beta_1 + \beta_2)\tau\cos 2\phi \tag{6.25}$$

from which we deduce that a longitudinal wave can propagate for $\tau \neq 0$ if either $\theta = \phi = \pi/4$ or $2\beta_1 + \beta_2 = 0$. The first of these possibilities corresponds to propagation along a principal axis and the second to a special set of values of the material parameters that allows propagation of a longitudinal and transverse wave in any in-plane direction.

In the linear specialization, we obtain

$$Q_{11} + Q_{22} = 3\alpha_1 + \alpha_2 + 2(1 + 2\beta_2 + \beta_2)\tau\sin 2\theta, \tag{6.26}$$

$$(Q_{11} - Q_{22})^2 + 4Q_{12}^2 = (\alpha_1 + \alpha_2)^2 + 4(\alpha_1 + \alpha_2)(2\beta_1 + \beta_2)\tau\sin 2\theta, \tag{6.27}$$

and the wave speeds (6.24) are then given by

$$\rho v^2 = \lambda_0 + 2\mu_0 + \{2\alpha_1'(0) + \alpha_2'(0) + [1 + 4\beta_1(0) + 2\beta_2(0)]\sin 2\theta\}\tau, \tag{6.28}$$

$$\rho v^2 = \mu_0 + [\alpha_1'(0) + \sin 2\theta]\tau \tag{6.29}$$

in which the coefficients have now been linearized in τ. These are respectively longitudinal and transverse when $\theta = \pi/4$, as indicated above, or in the special case, $2\beta_1 + \beta_2 = 0$ the transverse wave speed stands but the longitudinal wave speed specializes accordingly.

Tang (1967) considered in-plane wave propagation for an initial shear stress in which the only non-zero components of the second Piola–Kirchhoff stress were $T_{12} = T_{21}$. As with the case of hydrostatic pressure discussed at the end of Section 6.1, Tang used an incorrect incremental form of the constitutive law. When linearized in the initial stress the results from (3.10) in his paper that parallel (6.28) and (6.29) can be shown to reduce, in the present notation, to

$$\rho v^2 = \lambda_0 + 2\mu_0 + (\lambda_0 + 3\mu_0)\bar{\tau}\sin 2\theta, \quad \rho v^2 = \mu_0, \tag{6.30}$$

where we have set $\bar{\tau} = \tau/\mu_0$ and $\tau = T_{12}$ since there is no distinction between stress measures themselves in the reference configuration, which is quite different from the situation for their increments. Note, in particular, that $(6.30)_2$ depends on neither the initial shear stress nor the direction of propagation, which is quite unrealistic and cannot be recovered from (6.29) for $\tau \neq 0$. Fortuitously, $(6.30)_1$ can be recovered from (6.28) by making the special choices $2\alpha_1'(0) + \alpha_2'(0) = 0$ and $2\beta_1(0) + \beta_2(0) = 1 + \lambda_0/2\mu_0$ of the coefficients.

6.4 *Some connections with the classical theory of Biot*

It is interesting now to consider how the classical theory of initial stress in the general linear theory of elasticity due to Biot (see Biot, 1939, 1940a, 1965) relates to the present theory. As noted in Section 1, the elasticity tensor of Biot, with components $\mathcal{B}_{piqj}$ depends in an unspecified way on the initially stressed configuration. First, we record that, as shown in Ogden & Singh (2011), the general connection between $\mathcal{A}_{0piqj}$ and $\mathcal{B}_{piqj}$ may be written in the form

$$\mathcal{A}_{0piqj} = \mathcal{B}_{piqj} - \frac{1}{2}\delta_{pj}\tau_{qi} - \frac{1}{2}\delta_{pq}\tau_{ij} - \frac{1}{2}\delta_{qi}\tau_{pj} + \frac{1}{2}\delta_{ij}\tau_{pq} + \delta_{qj}\tau_{pi}, \tag{6.31}$$

which can be shown to be equivalent to the expression (4.25) given in Chapter 2 of Biot's book (Biot, 1965). For the general expression (4.2) to reduce to the Biot form for isotropic response, as quantified in (1.3), the material parameters in (4.2) must be specialized to

$$\alpha_1 = \mu_0, \quad \alpha_2 = \lambda_0, \quad \beta_1 = -1/2, \quad \beta_2 = 0, \tag{6.32}$$

where μ_0 and λ_0 are the classical Lamé moduli and terms of order higher than 1 in $\boldsymbol{\tau}$ are neglected.

For the hydrostatic stress considered in Section 6.1, these specializations yield, from (6.12) and (6.11), $\rho v_{\mathrm{T}}^2 = \mu_0$ and $\rho v_{\mathrm{L}}^2 = \lambda_0 + 2\mu_0 - \tau$. The first of these agrees with the original result of Biot, who mentioned that any dependence on the initial stress must be through the elastic constants themselves. The results in the present paper make the dependence explicit. It does not appear that the result $\rho v_{\mathrm{L}}^2 = \lambda_0 + 2\mu_0 - \tau$ was given by Biot.

We remark that in Shams *et al.* (2011), we adopted a slightly different form of the isotropic constitutive law from that given in (1.3), namely

$$\mathcal{B}_{piqj} = \mu_0(\delta_{ij}\delta_{pq} + \delta_{qi}\delta_{pj}) + \lambda_0\delta_{pi}\delta_{qj} + \delta_{pi}\tau_{qj} \tag{6.33}$$

for which, in the list (6.32), $\beta_2 = 0$ is replaced by $\beta_2 = 1$.

Turning next to the case of uniaxial stress, we find from Section 6.2 first that for propagation in the direction of initial stress the formulas (6.17) and (6.18) reduce to $\rho v_{11}^2 = \lambda_0 + 2\mu_0 - \tau$ and $\rho v_{12}^2 = \mu_0 + \frac{1}{2}\tau$, respectively, while for propagation normal to the initial stress the formulas (6.22) reduce to $\rho v_{22}^2 = \lambda_0 + 2\mu_0$, $\rho v_{21}^2 = \mu_0 - \frac{1}{2}\tau$ and $\rho v_{23}^2 = \mu_0$. For the case of compressive initial stress with $\tau = -P (P > 0)$, the speeds of the relevant two transverse waves agree with those obtained by Biot, specifically $\rho v_{\mathrm{T}}^2 = \mu_0 \pm \frac{1}{2}P$.

It is also interesting to apply the Biot specialization to the shear stress example in Section 6.3. Equations (6.28) and (6.29) yield

$$\rho v_{\mathrm{L}}^2 = \lambda_0 + 2\mu_0 - \tau \sin 2\theta, \quad \rho v_{\mathrm{T}}^2 = \mu_0 + \tau \sin 2\theta, \tag{6.34}$$

and a longitudinal and transverse wave can propagate in any in-plane direction, where θ is the angle the propagation direction makes with the principal direction of stress corresponding to principal stress $+\tau$. Note that for propagation along either principal direction, there is no influence of τ, which would seem to be unrealistic.

7. Deformed and pre-stressed isotropic elastic solid

In this section, we consider the initial stress to be associated with a finite deformation from an unstressed configuration (denoted $\mathcal{B}_0$) of an isotropic elastic solid, and we shall consider two states of the accompanying stress analogous to those considered in the foregoing sections, specifically pure dilatation, corresponding to hydrostatic stress, and an axially symmetric deformation corresponding to uniaxial stress. There is no direct analogue of the planar shear stress situation since when accompanied by deformation such as simple shear, there will in general also be normal stresses, which are not considered in Section 4.3.

For an isotropic elastic solid, the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by the appropriate specialization of (2.14), namely

$$J\boldsymbol{\sigma} = 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + 2I_3W_3\mathbf{I}, \tag{7.1}$$

where we recall that $J = \det \mathbf{F}$ and $\mathbf{B}$ is the left Cauchy–Green tensor, where $\mathbf{F}$ and J are now measured relative to $\mathcal{B}_0$ instead of $\mathcal{B}_\mathrm{r}$. We may consider the strain energy to depend on the three principal stretches $\lambda_1, \lambda_2, \lambda_3$ instead of the principal invariants I_1, I_2, I_3, and for this purpose, we write $W = \bar{W}(\lambda_1, \lambda_2, \lambda_3)$. The principal Cauchy stresses $\sigma_1, \sigma_2, \sigma_3$ are then given simply by

$$J\sigma_1 = \lambda_1\bar{W}_1, \quad J\sigma_2 = \lambda_2\bar{W}_2, \quad J\sigma_3 = \lambda_3\bar{W}_3, \tag{7.2}$$

which are equivalent to the principal components of (7.1), where $\bar{W}_i = \partial\bar{W}/\partial\lambda_i, i = 1, 2, 3$, and $J = \lambda_1\lambda_2\lambda_3$. This is easily seen by noting that in terms of the principal stretches the invariants I_1, I_2, I_3 are given by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{7.3}$$

In terms of invariants, the components of the elasticity tensor are given by

$$\begin{aligned} J\mathcal{A}_{0piqj} = {} & 2(W_1 + I_1W_2)B_{pq}\delta_{ij} + 2W_2[2B_{pi}B_{qj} - B_{iq}B_{jp} - B_{pr}B_{rq}\delta_{ij} - B_{pq}B_{ij}] \\ & + 2I_3W_3(2\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) + 4W_{11}B_{ip}B_{jq} + 4W_{22}(I_1B_{ip} - B_{ir}B_{rp})(I_1B_{jq} - B_{js}B_{sq}) \\ & + 4W_{12}(2I_1B_{ip}B_{jq} - B_{ip}B_{jr}B_{rq} - B_{jq}B_{ir}B_{rp}) + 4I_3W_{13}(B_{ip}\delta_{jq} + B_{jq}\delta_{ip}) \\ & + 4I_3W_{23}[I_1(B_{ip}\delta_{jq} + B_{jq}\delta_{ip}) - \delta_{ip}B_{jr}B_{rq} - \delta_{jq}B_{ir}B_{rp}] + 4I_3^2W_{33}\delta_{ip}\delta_{jq}, \end{aligned} \tag{7.4}$$

where B_{ij} are the components of $\mathbf{B}$. This specializes the form of $J\mathcal{A}_{0piqj}$ for an initially stressed solid given in Shams *et al.* (2011) to the present situation, but equivalent forms of (7.4) can be found in the earlier literature on isotropic finite elasticity (see, for example, Hayes & Rivlin, 1961).

When referred to the principal axes of $\mathbf{B}$ the components (7.4) can be expressed more compactly as

$$J\mathcal{A}_{0iijj} = \lambda_i\lambda_j\bar{W}_{ij}, \quad J\mathcal{A}_{0ijij} = \frac{\lambda_i\bar{W}_i - \lambda_j\bar{W}_j}{\lambda_i^2 - \lambda_j^2}\lambda_i^2, \quad J\mathcal{A}_{0ijji} = J\mathcal{A}_{0ijij} - \lambda_i\bar{W}_i, \tag{7.5}$$

where $\bar{W}_{ij} = \partial^2\bar{W}/\partial\lambda_i\partial\lambda_j, i, j \in \{1, 2, 3\}$. Note that the only non-zero components of $\boldsymbol{\mathcal{A}}_0$ are $\mathcal{A}_{0iijj}, i, j \in \{1, 2, 3\}$, together with, for $i \neq j$, $\mathcal{A}_{0ijij}$ and $\mathcal{A}_{0ijji}$ (see, for example, Ogden, 1984), as is the case for the components of the elasticity tensor of an initially stressed but undeformed material when referred to the principal axes of the initial stress $\boldsymbol{\tau}$, as can be seen by reference to (4.4).

If two of the principal stretches coincide, say $\lambda_j = \lambda_i$, then $\sigma_j = \sigma_i$ and a limiting process can be applied to express the second and third elements in (7.5) as

$$\mathcal{A}_{0ijij} \to \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + \sigma_i), \quad \mathcal{A}_{0ijji} \to \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} - \sigma_i), \tag{7.6}$$

respectively.

7.1 *Pure dilatation*

Here, we consider a pure dilatation with corresponding isotropic stress. Let $\lambda_i = J^{1/3}, i = 1, 2, 3$, and define $\hat{W}(J) = \bar{W}\big(J^{1/3}, J^{1/3}, J^{1/3}\big)$. We denote the corresponding (equal) principal stresses by σ. Then the simple connection $\sigma = \hat{W}'(J)$ follows. The only independent components of $\boldsymbol{\mathcal{A}}_0$ are then $\mathcal{A}_{0iijj} = J^{-1/3}\hat{W}_{ij}$ evaluated for the pure dilatation, with $\mathcal{A}_{0iiii}$ independent of i and $\mathcal{A}_{0iijj}$ independent of i and $j \neq i$. Now, from (5.1), we obtain

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \sigma\mathbf{L} - \sigma(\mathrm{tr}\,\mathbf{L})\mathbf{I}. \tag{7.7}$$

We consider an incremental simple shear deformation, and, without loss of generality, we may take this to be in the (x_1, x_2) plane. Then

$$\dot{\sigma}_{12} = \mathcal{A}_{01212}(L_{21} + L_{12}) = \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + \sigma)(L_{21} + L_{12}). \tag{7.8}$$

This enables us to define the incremental shear modulus as a function of J, which we write as $\hat{\mu}(J)$. It is given by

$$\hat{\mu}(J) = \frac{1}{2}(\mathcal{A}_{01111} - \mathcal{A}_{01122} + \sigma). \tag{7.9}$$

Note that $\dot{S}_{021} = \mathcal{A}_{02121}L_{12}$ for simple shear in the x_1 direction, $\dot{S}_{012} = \mathcal{A}_{01212}L_{21}$ for simple shear in the x_2 direction and $\mathcal{A}_{02121} = \mathcal{A}_{01212}$. We note in passing that $\dot{T}_{012} = (\mathcal{A}_{01212} - \sigma)(L_{21} + L_{12})$ and it would be incorrect to define the shear modulus as $\mathcal{A}_{01212} - \sigma$ based on use of $\dot{\mathbf{T}}_0$.

Next consider an incremental pure dilation ε, so that $L_{11} = L_{22} = L_{33} = \varepsilon/3$ and

$$\mathrm{tr}\dot{\boldsymbol{\sigma}} = (\mathcal{A}_{01111} + 2\mathcal{A}_{01122} - 2\sigma)\varepsilon. \tag{7.10}$$

The incremental bulk modulus, which we denote by $\hat{\kappa}(J)$, may be defined as a function of J as

$$\hat{\kappa}(J) = \frac{1}{3}(\mathcal{A}_{01111} + 2\mathcal{A}_{01122} - 2\sigma), \tag{7.11}$$

where the components of $\boldsymbol{\mathcal{A}}_0$ are again evaluated for $\lambda_i = J^{1/3}, i = 1, 2, 3$. This may also be expressed in the form $\hat{\kappa}(J) = J\hat{W}''(J)$, which agrees with a definition of the incremental bulk modulus adopted by Scott (2007).

If we set $\sigma = \tau$, where τ is the initial hydrostatic stress considered previously, then, because of the connection $\tau = \hat{W}'(J)$, we can in principle switch between the two different formulations, although, for a given τ, this would involve inversion of the relation $\tau = \hat{W}'(J)$ to obtain J. When the switch is made, we can identify $\hat{\mu}(J)$ and $\hat{\kappa}(J)$ with $\mu(\tau)$ and $\kappa(\tau)$, respectively. We should note here that the densities in the stress-free reference configuration $\mathcal{B}_0$, with density ρ_0, and the deformed (or initially stressed) configuration $\mathcal{B} = \mathcal{B}_\mathrm{r}$ are related by $\rho_0 = \rho J$ and the factor J needs to be used to switch between $\rho_0 v^2$ and ρv^2 in considering formulas for various wave speeds, where $\rho = \rho_\mathrm{r}$.

7.2 *Uniaxial stretch with lateral contraction*

Now consider a uniaxial stress $\sigma_1 = \sigma$ with $\sigma_2 = \sigma_3 = 0$ and stretches λ_1 and, by symmetry, $\lambda_2 = \lambda_3$. Then, $\bar{W}_2 = \bar{W}_3 = 0$ and the components of $\boldsymbol{\mathcal{A}}_0$ are given by

$$J\mathcal{A}_{01111} = \lambda_1^2 \bar{W}_{11}, \quad J\mathcal{A}_{02222} = \lambda_2^2 \bar{W}_{22} = J\mathcal{A}_{03333}, \quad J\mathcal{A}_{02233} = \lambda_2^2 \bar{W}_{23}, \tag{7.12}$$

$$J\mathcal{A}_{011ii} = \lambda_1\lambda_2 \bar{W}_{12}, \quad J\mathcal{A}_{01i1i} = \frac{\lambda_1^3 \bar{W}_1}{\lambda_1^2 - \lambda_2^2}, \quad J\mathcal{A}_{0i1i1} = \frac{\lambda_1\lambda_2^2 \bar{W}_1}{\lambda_1^2 - \lambda_2^2}, \quad i = 2, 3, \tag{7.13}$$

$$\mathcal{A}_{01ii1} = \mathcal{A}_{0i11i} = \mathcal{A}_{0i1i1} = \mathcal{A}_{01i1i} - \sigma, \quad i = 2, 3, \tag{7.14}$$

$$\mathcal{A}_{02323} = \mathcal{A}_{03232} = \mathcal{A}_{02332} = \mathcal{A}_{03223} = \frac{1}{2}(\mathcal{A}_{02222} - \mathcal{A}_{02233}), \tag{7.15}$$

all evaluated for $\lambda_3 = \lambda_2$ and with $\sigma = \lambda_2^{-2}\bar{W}_1$. Poisson's ratios and Young's moduli can be defined in exactly the same way as in Section 5.2 except that here the components of $\boldsymbol{\mathcal{A}}_0$ are different from those in Section 5.2. There is no need to repeat them all here, but we note, for example, that

$$\nu_{12} = \mathcal{A}_{01122}/(\mathcal{A}_{02222} + \mathcal{A}_{02233}) = \lambda_1 \bar{W}_{12}/\lambda_2(\bar{W}_{22} + \bar{W}_{23}), \tag{7.16}$$

$$E_1 = \mathcal{A}_{01111} - 2\nu_{12}\mathcal{A}_{01122} = J^{-1}\lambda_1^2[\bar{W}_{11} - 2\bar{W}_{12}^2/(\bar{W}_{22} + \bar{W}_{23})], \tag{7.17}$$

which are both functions of λ_1 when $\lambda_2 = \lambda_3$ is determined from $\bar{W}_2 = 0$ for a given form of $\bar{W}$. The expression for ν_{12} is consistent with the definition of the incremental Poisson's ratio given by Scott (2007). On the other hand, the definition of E_1 above differs from the corresponding definition in Scott (2007) since the latter is defined, in the present notation, as $\lambda_1 \mathrm{d}\sigma/\mathrm{d}\lambda_1$ with $\lambda_2 = \lambda_3$ given by $\bar{W}_2 = 0$. The definition of E_1 above corresponds to $\lambda_1 \mathrm{d}\bar{W}_1/\mathrm{d}\lambda_1$ updated to the deformed configuration by the push-forward factor $J^{-1}\lambda_1$, which is the appropriate specialization of the general push forward operation $J^{-1}\mathbf{F}$ that takes the nominal stress $\mathbf{S}$ to the Cauchy stress $\boldsymbol{\sigma} = J^{-1}\mathbf{FS}$. This definition of E_1 based on nominal stress is consistent with that used in Section 5.2.

7.3 *Application to second-order elasticity*

For definiteness, we now specialize the form of strain-energy function W to the third order in the strain. The precise form of this approximation depends on the choice of strain measure, but here, in order to make contact with several contributions to the literature, we shall use the Green strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, where $\mathbf{C}$ is again the right Cauchy–Green deformation tensor. The aim here is to obtain the first-order correction to the classical longitudinal and transverse wave speeds. For a discussion of the advantages of using logarithmic strain instead of Green strain, particularly when approaching the incompressible limit, we refer to the recent paper by Destrade & Ogden (2010).

7.3.1 *Historical expansions of the strain energy.* The use of invariants for expressing the third-order expansion of W appears to have been introduced by Brillouin (1925) (see also his monograph Brillouin, 1946). The Brillouin expansion may be written in the form

$$W = W_0 - \frac{1}{2}p_0\tilde{I}_1 + \frac{1}{8}\lambda_0\tilde{I}_1^2 + \frac{1}{4}\mu_0\tilde{I}_2 + A\tilde{I}_1\tilde{I}_2 + B\tilde{I}_1^3 + C\tilde{I}_3, \tag{7.18}$$

where, in different notation from that used by Brillouin, $\tilde{I}_1 = 2\mathrm{tr}\mathbf{E}$, $\tilde{I}_2 = 4\mathrm{tr}(\mathbf{E}^2)$, $\tilde{I}_3 = 8\mathrm{tr}(\mathbf{E}^3)$, respectively of orders 1, 2, 3 in $\mathbf{E}$, the constant W_0 is the energy in the reference configuration and p_0

corresponds to an initial pressure in the reference configuration. Note, in particular, that Brillouin used $2\mathbf{E}$ rather than $\mathbf{E}$ itself as the strain measure and that the notations A, B, C are different from those defined in (6.14).

Let us drop the terms W_0 and p_0, which are redundant for our purposes, and recast the remaining terms using the principal invariants of Green strain, which, for consistency with the notation used in Destrade & Ogden (2010), we denote by i_1, i_2, i_3. Thus,

$$i_1 = \mathrm{tr}\mathbf{E}, \quad i_2 = \frac{1}{2}[i_1^2 - \mathrm{tr}(\mathbf{E}^2)], \quad i_3 = \det\mathbf{E}. \tag{7.19}$$

Then, we have

$$W = \frac{1}{2}(\lambda_0 + 2\mu_0)i_1^2 - 2\mu_0 i_2 + 8(A + B + C)i_1^3 - 8(2A + 3C)i_1 i_2 + 24C i_3. \tag{7.20}$$

This is entirely equivalent to the strain-energy function generally referred to as the Murnaghan form of strain energy that appears in Murnaghan (1951, 1967) and is based on the use of Green strain. It is written

$$W = \frac{1}{2}(\lambda_0 + 2\mu_0)i_1^2 - 2\mu_0 i_2 + \frac{1}{3}(l + 2m)i_1^3 - 2m i_1 i_2 + n i_3, \tag{7.21}$$

where l, m, n are the Murnaghan constants. The Brillouin constants A, B, C are given in terms of l, m, n by

$$A = \frac{1}{16}(2m - n), \quad B = \frac{1}{48}(2l - 2m + n), \quad C = \frac{1}{24}n. \tag{7.22}$$

We remark that in his original paper, Murnaghan (1937) worked in terms of the Almansi strain tensor $(\mathbf{I}-\mathbf{B}^{-1})/2$ and its principal invariants, which we denote here by $\bar{I}_1, \bar{I}_2, \bar{I}_3$. The original energy function of Murnaghan has the form

$$W = \frac{1}{2}(\lambda_0 + 2\mu_0)\bar{I}_1^2 - 2\mu_0 \bar{I}_2 + \bar{l}\bar{I}_1^3 + \bar{m}\bar{I}_1\bar{I}_2 + \bar{n}\bar{I}_3. \tag{7.23}$$

In general, (7.23) is different from (7.21), but the two are equivalent to the third order in the strains. It is then easy to show that the constants l, m, n and $\bar{l}, \bar{m}, \bar{n}$ are related by

$$\bar{l} = \frac{1}{3}(l + 2\lambda_0), \quad \bar{m} = -2m - 4\lambda_0 - 12\mu_0, \quad \bar{n} = n + 12\mu_0. \tag{7.24}$$

This difference has significance when considering approximations to wave speeds at this order as we shall see shortly. We note in passing that because of the requirement of objectivity the Almansi strain tensor (or any other Eulerian strain tensor) can be used as the argument of the strain energy function if and only if the material is isotropic. Clearly, the work of Brillouin on this topic has been to some extent overlooked, although Murnaghan (1937), Hughes & Kelly (1953) and Truesdell (1961) did refer to Brillouin (1925). For further historical discussion of second-order elasticity, including the contribution of Rivlin (1953), we refer to Section 66 of Truesdell & Noll (1965).

An equivalent form of the third-order expanded energy function was also introduced by Landau & Rumer (1937), who were apparently unaware of the work of Brillouin. This may be written

$$W = \frac{1}{2}\lambda_0(\mathrm{tr}\mathbf{E})^2 + \mu_0\mathrm{tr}(\mathbf{E}^2) + \frac{1}{3}\bar{A}\,\mathrm{tr}(\mathbf{E}^3) + \bar{B}(\mathrm{tr}\mathbf{E})\mathrm{tr}(\mathbf{E}^2) + \frac{1}{3}\bar{C}(\mathrm{tr}\mathbf{E})^3, \tag{7.25}$$

where overbars have been used to distinguish the third-order constants from those of Brillouin; in Landau & Lifshitz (1986), the notations A, B, C were used, differing from those in Landau & Rumer (1937). The connections between the Murnaghan constants l, m, n and the Landau constants $\bar{A}, \bar{B}, \bar{C}$ were noted in Destrade & Ogden (2010) as

$$\bar{A} = n, \quad \bar{B} = m - \frac{1}{2}n, \quad \bar{C} = l - m + \frac{1}{2}n. \tag{7.26}$$

Biot (1940b) also developed a third-order expansion, which is equivalent to the above and details may also be found in his book (Biot, 1965). Biot (1965) worked in terms of the principal strain components $\lambda_i - 1$ and the corresponding principal Biot stresses. His third-order constants, denoted D, F, G, can be shown to be related to l, m, n via

$$D = l + 2m + \frac{3}{2}(\lambda_0 + 2\mu_0), \quad F = l + \frac{1}{2}\lambda_0, \quad G = 2l - 2m + n. \tag{7.27}$$

He did not give the form of strain energy explicitly.

Finally, we mention the third-order expansion adopted by Toupin & Bernstein (1961), who used the invariants $\mathrm{tr}\,\mathbf{E}$, $\mathrm{tr}\,(\mathbf{E}^2)$, $\mathrm{tr}\,(\mathbf{E}^3)$ and third-order constants ν_1, ν_2, ν_3. In terms of the principal invariants of Green strain, their energy function has the form

$$W = \frac{1}{2}(\lambda_0 + 2\mu_0)i_1^2 - 2\mu_0 i_2 + \frac{1}{6}(\nu_1 + 6\nu_2 + 8\nu_3)i_1^3 - 2(\nu_2 + 2\nu_3)i_1 i_2 + 4\nu_3 i_3. \tag{7.28}$$

It is straightforward to show that the constants ν_1, ν_2, ν_3 are related to l, m, n and $\bar{A}, \bar{B}, \bar{C}$ by (Norris, 1998)

$$\nu_1 = 2l - 2m + n = 2\bar{C}, \quad \nu_2 = m - \frac{1}{2}n = \bar{B}, \quad \nu_3 = \frac{1}{4}n = \frac{1}{4}\bar{A}. \tag{7.29}$$

Connections between some of the above sets of constants and others used by Rivlin (1953) were also noted by Truesdell & Noll (1965).

Here, we shall work in terms of the Murnaghan constants l, m, n but cast (7.21) in terms of the principal invariants (2.9) of $\mathbf{C}$ as

$$\begin{aligned} W = {} & \frac{\lambda_0}{8}(I_1 - 3)^2 + \frac{\mu_0}{4}(I_1^2 - 2I_1 - 2I_2 + 3) \\ & + \frac{l}{24}(I_1 - 3)^3 + \frac{m}{12}(I_1 - 3)(I_1^2 - 3I_2) + \frac{n}{8}(I_1 - I_2 + I_3 - 1). \end{aligned} \tag{7.30}$$

For this energy function, we have $W_{22} = W_{13} = W_{23} = W_{33} = 0$ and the expression (7.4) reduces to

$$\begin{aligned} J\mathcal{A}_{0piqj} = {} & 2(W_1 + I_1 W_2)B_{pq}\delta_{ij} + 2W_2[2B_{pi}B_{qj} - B_{iq}B_{jp} - B_{pr}B_{rq}\delta_{ij} - B_{pq}B_{ij}] \\ & + 2I_3 W_3(2\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) + 4W_{11}B_{ip}B_{jq} \\ & + 4W_{12}(2I_1 B_{ip}B_{jq} - B_{ip}B_{jr}B_{rq} - B_{jq}B_{ir}B_{rp}), \end{aligned} \tag{7.31}$$

and the remaining coefficients W_1, W_2, W_3, W_{11} and W_{12} are simply obtained from (7.30).

In working with second-order elasticity, the corrections to the classical moduli are obtained by expanding the coefficients in the above to the first order in $\mathbf{E}$. We have $\mathbf{C} = \mathbf{I} + 2\mathbf{E}$ and $I_1 = 3 + 2E$, exactly, which we use together with the linear approximations $I_2 \simeq 3+4E$, $I_3 \simeq 1+2E$, $\mathbf{B} \simeq \mathbf{C} = \mathbf{I} + 2\mathbf{E}$, where $E = \mathrm{tr}\mathbf{E}$. We also note that $\rho \simeq \rho_{\mathrm{r}}(1-E)$. To the first order in $\mathbf{E}$, we obtain

$$\begin{aligned} W_1 &= \mu_0 + \frac{1}{8}n + \frac{1}{2}(\lambda_0 + 2\mu_0 + 2m)E, \quad W_2 = -\frac{1}{2}\mu_0 - \frac{1}{8}n - \frac{1}{2}mE, \quad W_3 = \frac{1}{8}n, \\ W_{11} &= \frac{1}{4}(\lambda_0 + 2\mu_0 + 4m) + \frac{1}{2}(l+2m)E, \quad W_{12} = -\frac{1}{4}m, \end{aligned} \tag{7.32}$$

and hence

$$\begin{aligned} J\mathcal{A}_{0piqj} \simeq\ & \mu_0(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}) + \lambda_0\delta_{ip}\delta_{jq} + \frac{1}{2}[(2\lambda_0 + 2m - n)\delta_{ij}\delta_{pq} + 2(2l - 2m + n)\delta_{ip}\delta_{jq} \\ & + (2m-n)\delta_{iq}\delta_{jp}]E + \frac{1}{2}(4\mu_0 + n)(\delta_{pq}E_{ij} + \delta_{iq}E_{jp} + \delta_{jp}E_{iq}) \\ & + \frac{1}{2}(8\mu_0 + n)\delta_{ij}E_{pq} + (2\lambda_0 + 2m - n)(\delta_{ip}E_{jq} + \delta_{jq}E_{ip}). \end{aligned} \tag{7.33}$$

We are particularly interested in the case of a pure dilatation, for which, with $E_{ij} = E\delta_{ij}/3$, (7.33) reduces to

$$\begin{aligned} J\mathcal{A}_{0piqj} =\ & \mu_0(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}) + \lambda_0\delta_{ip}\delta_{jq} + \frac{1}{6}(12\mu_0 + 6\lambda_0 + 6m - n)E\delta_{ij}\delta_{pq} \\ & + \frac{1}{6}(8\mu_0 + 6m - n)E\delta_{iq}\delta_{pj} + \frac{1}{3}(4\lambda_0 + 6l - 2m + n)E\delta_{pi}\delta_{qj}. \end{aligned} \tag{7.34}$$

From the definitions (7.9) and (7.11), we now obtain the approximations

$$\hat{\mu}(J) \simeq \mu_0 + \frac{1}{6}(6\lambda_0 + 6\mu_0 + 6m - n)E \tag{7.35}$$

and

$$\hat{\kappa}(J) \simeq \kappa_0 + \frac{2}{9}(9l + n)E, \tag{7.36}$$

for small dilatation, where to the first order $J = 1 + E$, and we note that $\hat{\mu}(1) = \mu_0$, $\hat{\kappa}(1) = \kappa_0$.

To the second order, we may expand the isotropic stress as

$$\sigma = \hat{W}'(J) \simeq \varepsilon\hat{W}''(1) + \frac{1}{2}\varepsilon^2\hat{W}'''(1), \tag{7.37}$$

where $\hat{W}''(1) = \kappa_0$, $\varepsilon \equiv J - 1 \simeq E + E^2/6$ and, for the Murnaghan strain energy, $\hat{W}'''(1) = -\kappa_0 + 2l + 2n/9$.

As indicated earlier, if the initial stress $\boldsymbol{\tau}$ discussed in Sections 4 and 5 is associated with a pure dilatation, so that $\boldsymbol{\tau} = \tau\mathbf{I}$, then in principle the results here can be converted to those based on τ. In particular, if we set $\sigma = \tau$ in the above and work to second order, then we may invert the $\tau \longleftrightarrow \varepsilon$ relation in the form

$$\varepsilon \simeq \tau/\kappa_0 - \frac{1}{2}\hat{W}'''(1)\tau^2/\kappa_0^3, \tag{7.38}$$

but in considering the linear approximations of the shear and bulk moduli only the first-order term need be retained. Then, with $\tau = \kappa_0\varepsilon$, we can identify $\mu(\tau)$ and $\kappa(\tau)$ with $\hat{\mu}(J)$ and $\hat{\kappa}(J)$, respectively. Thus,

$$\mu(\tau) \simeq \mu_0 + [\alpha_1'(0) + 2\beta_1(0) + 1]\kappa_0\varepsilon \simeq \hat{\mu}(J) \simeq \mu_0 + \frac{1}{6}(6\lambda_0 + 6\mu_0 + 6m - n)\varepsilon,$$
$$\kappa(\tau) \simeq \kappa_0 + \frac{1}{3}[2\alpha_1'(0) + 4\beta_1(0) + 3\alpha_2'(0) + 6\beta_2(0) - 1]\kappa_0\varepsilon \simeq \hat{\kappa}(J) \simeq \kappa_0 + \frac{2}{9}(9l + n)\varepsilon. \quad (7.39)$$

Hence, we can relate the constants $\alpha_1'(0)$, $\alpha_2'(0)$, $\beta_1(0)$, $\beta_2(0)$ to the Murnaghan constants. After a little rearrangement, this yields

$$\alpha_1'(0) + 2\beta_1(0) = (2\mu_0 + 6m - n)/6\kappa_0, \quad (7.40)$$

$$\alpha_2'(0) + 2\beta_2(0) = (\lambda_0 + 6l - 2m + n)/3\kappa_0. \quad (7.41)$$

In fact, the separate values of $\alpha_1'(0)$, $\alpha_2'(0)$, $\beta_1(0)$ and $\beta_2(0)$ can be obtained by considering other initial deformations than pure dilatation, and we show this in Section 7.3.3 by comparing wave speeds based on the theory of uniaxial initial stress from Section 6.2 with those based on a finite deformation from an isotropic reference configuration under uniaxial stress.

7.3.2 *Implications for the wave speeds.* Turning now to the propagation of plane waves, we note that for any (**m**, **n**) pair satisfying (6.5), the wave speed v is given by

$$\rho_0 v^2 = J[\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} = J\mathcal{A}_{0piqj} n_p n_q m_i m_i, \quad (7.42)$$

where the factor J is now included to reflect the change in reference configuration from $\mathcal{B}_0$ to $\mathcal{B}_r = \mathcal{B}$. For the case of pure dilatation, we then obtain, on use of (7.34),

$$\rho_0 v^2 = \mu_0 + (\mu_0 + \lambda_0)(\mathbf{m} \cdot \mathbf{n})^2 + \frac{1}{6}(12\mu_0 + 6\lambda_0 + 6m - n)E \quad (7.43)$$

$$+\frac{1}{6}[8\lambda_0 + 8\mu_0 + 12l + 2m + n]E(\mathbf{m} \cdot \mathbf{n})^2. \quad (7.44)$$

For a longitudinal wave with $\mathbf{m} = \mathbf{n}$, this reduces to

$$\rho_0 v_{\mathrm{L}}^2 = \lambda_0 + 2\mu_0 + \frac{1}{3}(7\lambda_0 + 10\mu_0 + 6l + 4m)E, \quad (7.45)$$

and for a transverse wave, with $\mathbf{m} \cdot \mathbf{n} = 0$

$$\rho_0 v_{\mathrm{T}}^2 = \mu_0 + \left(\lambda_0 + 2\mu_0 + m - \frac{1}{6}n\right)E. \quad (7.46)$$

These results agree with those obtained by Hughes & Kelly (1953) for the case of hydrostatic pressure ($\tau = -P$). They used the Murnaghan energy function based on Green strain. As shown by Shams *et al.* (2011), there is similar agreement in the case of the uniaxial compression considered by Hughes & Kelly (1953). Toupin & Bernstein (1961) obtained equivalent results based on their third-order expansion, but expressed in terms of the acoustoelastic coefficients, which we write as

$$\frac{\mathrm{d}}{\mathrm{d}E}(\rho_0 v_L^2)\Big|_{E=0} = \frac{1}{3}(7\lambda_0 + 10\mu_0 + 3\nu_1 + 10\nu_2 + 8\nu_3), \quad (7.47)$$

$$\frac{\mathrm{d}}{\mathrm{d}E}(\rho_0 v_T^2)\Big|_{E=0} = \frac{1}{3}(3\lambda_0 + 6\mu_0 + 3\nu_2 + 4\nu_3). \quad (7.48)$$

TABLE 3 *Lamé constants and Landau third-order elastic moduli for five solids (expressed in units of 10^9 $N m^{-2}$), as collected by Porubov (2003): his Murnaghan constants m, n, l have been converted here to the Toupin–Bernstein constants ν_1, ν_2, ν_3*

Material	λ_0	μ_0	ν_1	ν_2	ν_3
Polystyrene	1.71	−21.2	−10	−8.3	−2.5
Steel Hecla 37	−354	82.1	−358	−282	−89.5
Aluminium 2S	−204	27.6	−228	−197	−57
Pyrex glass	264	27.5	420	−118	105
SiO_2 melted	72	31.3	−44	93	−11

Note that the E in Toupin & Bernstein (1961) is 3× that used here. Toupin & Bernstein (1961) mentioned that their results were equivalent to those obtained by Brillouin (1925). The Brillouin results therefore pre-date much of the subsequent work.

In calculating the second-order correction to longitudinal and transverse wave speeds, Birch (1938) used the original (Almansi strain based) form of the Murnaghan strain energy but effectively set $\bar{l} = \bar{m} = \bar{n} = 0$, obtaining the results

$$\rho_0 v_L^2 = \lambda_0 + 2\mu_0 + P(13\lambda_0 + 14\mu_0)/(3\kappa_0) \equiv \lambda_0 + 2\mu_0 - (13\lambda_0 + 14\mu_0)E/3, \tag{7.49}$$

$$\rho_0 v_T^2 = \mu_0 + P(\lambda_0 + 2\mu_0)/\kappa_0 \equiv \mu_0 - (\lambda_0 + 2\mu_0)E \tag{7.50}$$

for the wave speeds, where $P = -\kappa_0 E$ is the pressure. The omission of the third-order constants can be significant since typically they are of the same order of magnitude as the Lamé moduli λ_0 and μ_0, as illustrated by the data shown in Table 3.

If the third-order constants are retained, then we have, instead,

$$\rho_0 v_L^2 = \lambda_0 + 2\mu_0 - (13\lambda_0 + 14\mu_0 - 18\bar{l} + 2\bar{m})E/3,$$
$$\rho_0 v_T^2 = \mu_0 - (\lambda_0 + 2\mu_0 + \frac{1}{2}\bar{m} + \frac{1}{6}\bar{n})E. \tag{7.51}$$

If, by contrast, we set the third-order constants l, m, n to zero, then the Hughes & Kelly (1953) results reduce to

$$\rho_0 v_L^2 = \lambda_0 + 2\mu_0 + (7\lambda_0 + 10\mu_0)E/3,$$
$$\rho_0 v_T^2 = \mu_0 + (\lambda_0 + 2\mu_0)E. \tag{7.52}$$

Note, in particular, the opposite sign but equal magnitude of the second term in the shear wave expression compared with (7.50). Thus, interpretation of the results requires caution. In particular, care must be taken that the results allow for an increase as well as for a decrease of the wave speeds with pressure and uniaxial stress, depending on which solid is considered (see Tables 1 and 2).

7.3.3 *The case of uniaxial stress.* We now consider a deformation from a stress-free configuration $\mathcal{B}_0$ of an isotropic material associated with a uniaxial stress σ in the $\mathbf{e}_1$ direction. We denote the corresponding component E_{11} of the Green strain tensor by E. Then, by setting the lateral stress to zero and by symmetry, $E_{22} = E_{33} = -\lambda_0 E/2(\lambda_0 + \mu_0)$, to the first order in E, and $\sigma = 3\kappa_0\mu_0 E/(\lambda_0 + \mu_0)$, and hence

$$J = 1 + \mu_0 E/(\lambda_0 + \mu_0) = 1 + \sigma/3\kappa_0, \tag{7.53}$$

also to first order.

The wave speeds $v_{11}, v_{12}, v_{22}, v_{23}$ are then calculated by using (7.33) and appropriate specializations of (7.42) and the connection $\rho_0 = \rho J$. After some manipulations, which are omitted, this yields the formulas

$$\rho v_{11}^2 = \lambda_0 + 2\mu_0 + 2[2\lambda_0^2 + 7\lambda_0\mu_0 + 4\mu_0^2 + \mu_0 l + 2(\lambda_0 + \mu_0)m]\sigma/3\kappa_0\mu_0, \tag{7.54}$$

$$\rho v_{12}^2 = \mu_0 + [4\mu_0(4\lambda_0 + 3\mu_0) + 4\mu_0 m + \lambda_0 n]\sigma/12\kappa_0\mu_0, \tag{7.55}$$

$$\rho v_{22}^2 = \lambda_0 + 2\mu_0 - (2\lambda_0^2 + 5\lambda_0\mu_0 + 2\mu_0^2 - 2\mu_0 l + 2\lambda_0 m)\sigma/3\kappa_0\mu_0, \tag{7.56}$$

$$\rho v_{23}^2 = \mu_0 - [2\mu_0(2\lambda_0 + \mu_0) - 2\mu_0 m + (\lambda_0 + \mu_0)n]\sigma/6\kappa_0\mu_0, \tag{7.57}$$

and we also note the connection $\rho v_{12}^2 - \rho v_{21}^2 = \sigma$. By setting $\sigma = \tau$, we then compare these results with the formulas in (6.17), (6.18) and (6.22), which we now collect together as

$$\rho v_{11}^2 = \lambda_0 + 2\mu_0 + [2\bar{\alpha}_1'(0) + \bar{\alpha}_2'(0) + 4\beta_1(0) + 2\beta_2(0) + 1]\tau, \tag{7.58}$$

$$\rho v_{12}^2 = \mu_0 + [\bar{\alpha}_1'(0) + \beta_1(0) + 1]\tau, \tag{7.59}$$

$$\rho v_{22}^2 = \lambda_0 + 2\mu_0 + [2\bar{\alpha}_1'(0) + \bar{\alpha}_2'(0)]\tau, \tag{7.60}$$

$$\rho v_{23}^2 = \mu_0 + \bar{\alpha}_1'(0)\tau, \tag{7.61}$$

with ρv_{21}^2 given by (6.21). Note, in particular, that bars have now been placed over $\alpha_1'(0)$ and $\alpha_2'(0)$. This is because the arguments of α_1 and α_2 are different for hydrostatic stress and uniaxial stress, respectively, the relevant invariants of $\boldsymbol{\tau}$ are $(3\tau, 3\tau^2, 3\tau^3)$ and (τ, τ^2, τ^3), so that $\alpha_1'(0) = 3\bar{\alpha}_1'(0)$ and $\alpha_2'(0) = 3\bar{\alpha}_2'(0)$, while $\beta_1(0)$ and $\beta_2(0)$ are the same in each case.

Comparison of the two sets of formulas yields the results

$$\beta_1(0) = 1 + n/4\mu_0, \quad \beta_2(0) = (2\lambda_0 + 2m - n)/2\mu_0, \tag{7.62}$$

$$\alpha_1'(0) = -[2(2\lambda_0 + \mu_0)\mu_0 - 2\mu_0 m + (\lambda_0 + \mu_0)n]/2\kappa_0\mu_0, \tag{7.63}$$

$$\alpha_2'(0) = -[(2\lambda_0 + \mu_0)\lambda_0 - 2\mu_0 l + (2m - n)(\lambda_0 + \mu_0)]/\kappa_0\mu_0 \tag{7.64}$$

from which it is easy to check that the results (7.40) and (7.41) are recovered.

It is interesting that the *four* constants $\alpha_1'(0)$, $\alpha_2'(0)$, $\beta_1(0)$ and $\beta_2(0)$ are expressed in terms of the *three* Murnaghan constants. This is explained by noting that the anisotropic constitutive law for an initially stressed material with no accompanying deformation is specialized to isotropy by introducing a stress-free reference configuration and an associated deformation.

Finally, we note that no choice of the Murnaghan constants will give the Biot values (6.32) with $\alpha_1'(0) = \alpha_2'(0) = 0$.

8. Concluding remarks

In this paper, we have examined in detail the effect of initial stress on the propagation of small amplitude homogeneous plane waves in an undeformed elastic material on the basis of the general theory of a hyperelastic material with initial stress developed by Shams *et al.* (2011), which had its genesis in the work of Hoger (1985, 1986, 1993a,b) in particular. A key feature of the constitutive law, formulated in terms of invariants of the deformation and initial stress, is that the elasticity tensor depends in general in a highly non-linear way on initial stress. Important special cases considered within the general framework included initial stresses corresponding to hydrostatic stress, uniaxial stress and shear stress for which explicit and relatively simple forms of the elasticity tensor were given. For each of these states of stress, the dependence of various elastic moduli on the initial stress was made explicit. For example, simple formulas were obtained for the stress dependence of the Lamé moduli in the case of isotropic initial stress and Poisson's ratios and Young's moduli for the cases of uniaxial initial stress and planar initial shear stress.

The results were applied to infinitesimal wave propagation and it was shown how some known results fit within the general framework, and some discrepancies in some of the earlier work were highlighted. We then considered the initial stress to be a pre-stress associated with the deformation of an isotropic elastic material from a stress-free reference configuration and made connections with the analysis from the preceding sections. Specifically, we considered a pure dilatational deformation and a deformation corresponding to simple tension. We then discussed the specialization of second-order elasticity in detail and collated various contributions from the earlier literature that date back to the work of Brillouin (1925), with particular reference to expressions for longitudinal and transverse wave speeds, again showing how the results are captured within the general framework herein.

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Appendix A

Non-principal waves in a solid under initial uniaxial stress

Here, we complete the analysis of Section 6.2 by considering the case of non-principal wave propagation for which the direction of propagation $\mathbf{n}$ and the direction of uniaxial stress $\mathbf{a}$ are neither parallel nor orthogonal. The constants A, B, C and D are as defined in (6.14).

By solving (6.6) with $\mathbf{Q}(\mathbf{n})$ given by (6.13), we find that the wave speeds are given by

$$\rho v^2 = A, \quad (A - \rho v^2)^2 + [B + C + 2D(\mathbf{n}\cdot\mathbf{a})](A - \rho v^2) + (BC - D^2)[1 - (\mathbf{n}\cdot\mathbf{a})^2] = 0. \quad \text{(A.1)}$$

We therefore consider the cases $\rho v^2 = A$ and $\rho v^2 \neq A$ separately.

Case 1. $\rho v^2 = A$.

The propagation condition (6.5) yields

$$[B(\mathbf{m}\cdot\mathbf{a}) + D(\mathbf{m}\cdot\mathbf{n})]\mathbf{a} + [C(\mathbf{m}\cdot\mathbf{n}) + D(\mathbf{m}\cdot\mathbf{a})]\mathbf{n} = \mathbf{0}. \quad \text{(A.2)}$$

If $C(\mathbf{m}\cdot\mathbf{n}) + D(\mathbf{m}\cdot\mathbf{a}) \neq 0$, then $\mathbf{n} = \pm\mathbf{a}$ and hence $(B + C \pm 2D)(\mathbf{m}\cdot\mathbf{a}) = 0$. The case $\mathbf{m}\cdot\mathbf{a} = 0$ was covered in Section 6.2, but there is now an additional possibility that $B + C \pm 2D = 0$. Both these options lead to the same result, which, on substitution from (6.14), is written

$$\alpha_1 + \alpha_2 + (3\beta_1 + 2\beta_2)\tau + (3\gamma_1 + 2\gamma_2 + \beta_3)\tau^2 + 2\gamma_3\tau^3 + \gamma_4\tau^4 = 0. \quad \text{(A.3)}$$

There is no restriction on the direction of polarization $\mathbf{m}$. Note that for the specialization (6.32), this yields $\tau = -2(\lambda_0 + \mu_0)$ and $A = -\lambda_0$ and for several of the values of λ_0 listed in Table 3, there is no real wave speed in this case.

Next, consider the possibility that $\mathbf{n} \neq \pm\mathbf{a}$. Then, if $C(\mathbf{m}\cdot\mathbf{n}) + D(\mathbf{m}\cdot\mathbf{a}) = 0$, it follows that also $B(\mathbf{m}\cdot\mathbf{a}) + D(\mathbf{m}\cdot\mathbf{n}) = 0$. By combining these, we deduce that

$$(BC - D^2)(\mathbf{m}\cdot\mathbf{a}) = 0, \quad (BC - D^2)(\mathbf{m}\cdot\mathbf{n}) = 0, \quad \text{(A.4)}$$

provided $C \neq 0$, $D \neq 0$. Then, if $BC - D^2 \neq 0$, we must have $\mathbf{m} \cdot \mathbf{a} = 0$ and $\mathbf{m} \cdot \mathbf{n} = 0$ and $\mathbf{m}$ is normal to the plane of $\mathbf{a}$ and $\mathbf{n}$. Thus, a transverse wave exists for any direction of propagation. On the other hand, if $BC - D^2 = 0$, then $\mathbf{n}$ is determined from the equation

$$BC - D^2 \equiv (\alpha_1 + \alpha_2)\{\beta_1\tau + \gamma_1\tau^2 + [\beta_3 + 2\gamma_3\tau + \gamma_4\tau^2]\tau^2(\mathbf{n}\cdot\mathbf{a})^2\} - [\beta_1 + \beta_2 + (\gamma_1 + \gamma_2)\tau]^2\tau^2(\mathbf{n}\cdot\mathbf{a})^2 = 0. \quad \text{(A.5)}$$

Since we are considering the case $\mathbf{n} \neq \pm\mathbf{a}$ and $\mathbf{n} \cdot \mathbf{a} \neq 0$, possible directions $\mathbf{n}$ generate a cone with axis $\mathbf{a}$, provided $|\mathbf{n} \cdot \mathbf{a}| < 1$. For each such $\mathbf{n}$, $\mathbf{m}$ must satisfy $\mathbf{m} \cdot (B\mathbf{a} + D\mathbf{n}) = 0$. Note that in the linear approximation (A.5) cannot hold unless $\beta_1(0) = 0$ in which case $\mathbf{n}$ is unrestricted and $A = \mu_0 + \alpha_1'(0)\tau + \tau(\mathbf{n}\cdot\mathbf{a})^2$.

Other special cases are as follows: if $B \neq 0$, $C \neq 0$, $D = 0$, then $\mathbf{m} \cdot \mathbf{n} = 0$, $\mathbf{m} \cdot \mathbf{a} = 0$ and either $\mathbf{n} \cdot \mathbf{a} = 0$ or $\beta_1 + \beta_2 + (\gamma_1 + \gamma_2)\tau = 0$; if $B = 0$, $C \neq 0$, $D = 0$, then $\mathbf{m} \cdot \mathbf{n} = 0$ and either $\mathbf{n} \cdot \mathbf{a} = 0$ or $\beta_1 + \beta_2 + (\gamma_1 + \gamma_2)\tau = 0$. In the latter, if $B = 0$ and $\mathbf{n} \cdot \mathbf{a} = 0$, then $\beta_1 + \gamma_1\tau = 0$ (assuming, of course, $\tau \neq 0$), while if $B = 0$ and $\mathbf{n} \cdot \mathbf{a} \neq 0$, then $(\mathbf{n} \cdot \mathbf{a})^2$ is determined from $B = 0$. Finally, we note that if $C = D = 0$, then there are four possibilities: (i) $\mathbf{n} \cdot \mathbf{a} = 0$ and $B = 0$ and hence $\beta_1 + \gamma_1\tau = 0$—there is no restriction on $\mathbf{m}$; (ii) $\mathbf{n} \cdot \mathbf{a} = 0$ and $\mathbf{m} \cdot \mathbf{a} = 0$—this is captured by the discussion around (6.19); (iii) $\beta_1 + \beta_2 + (\gamma_1 + \gamma_2)\tau = 0$ and $B = 0$, the latter determining $(\mathbf{n} \cdot \mathbf{a})^2$—there is no restriction on $\mathbf{m}$ and (iv) $\beta_1 + \beta_2 + (\gamma_1 + \gamma_2)\tau = 0$ and $\mathbf{m} \cdot \mathbf{a} = 0$—there is no restriction on $\mathbf{n}$.

Case 2. $\rho v^2 \neq A$.

Now, from the propagation condition (6.5) with the specialization (6.13), we have

$$(A - \rho v^2)\mathbf{m} + [B(\mathbf{m}\cdot\mathbf{a}) + D(\mathbf{m}\cdot\mathbf{n})]\mathbf{a} + [C(\mathbf{m}\cdot\mathbf{n}) + D(\mathbf{m}\cdot\mathbf{a})]\mathbf{n} = \mathbf{0}. \quad \text{(A.6)}$$

Thus, $\mathbf{m}$, $\mathbf{n}$, $\mathbf{a}$ are coplanar unless either the coefficient of $\mathbf{a}$ or $\mathbf{n}$ vanishes. If neither of the coefficients vanish, then, without loss of generality, we may confine attention to the (x_1, x_2) plane and set $\mathbf{a} = \mathbf{e}_1$. Let (n_1, n_2) and (m_1, m_2) be the in-plane components of $\mathbf{n}$ and $\mathbf{m}$, respectively, and set $n_3 = m_3 = 0$. The propagation condition (6.5) then specializes to the two components

$$Q_{11}m_1 + Q_{12}m_2 = \rho v^2 m_1, \quad Q_{12}m_1 + Q_{22}m_2 = \rho v^2 m_2. \quad \text{(A.7)}$$

For a given propagation direction, the wave speed is given by one of the two solutions of the quadratic in (A.1) and is known explicitly. Elimination of ρv^2 from (A.7) then gives

$$(Q_{11} - Q_{22})m_1 m_2 = Q_{12}(m_1^2 - m_2^2), \quad \text{(A.8)}$$

which determines the polarization $\mathbf{m}$. Let $n_1 = \cos\theta$, $n_2 = \sin\theta$ and $m_1 = \cos\phi$, $m_2 = \sin\phi$. Then the above can be rewritten as

$$\tan 2\phi = \frac{C\sin 2\theta + 2D\sin\theta}{B + C\cos 2\theta + 2D\cos\theta}. \quad \text{(A.9)}$$

From this, we can immediately recover some of the previous results. If, for example, $\theta = 0$ (propagation in the direction of initial stress), then $\phi = 0$ or $\pi/2$, corresponding to longitudinal and transverse waves, respectively, with wave speeds given by $\rho v_{\mathrm{L}}^2 = A + B + C + 2D$ and $\rho v_{\mathrm{T}}^2 = A$ (degenerate case). If $\theta = \pi/2$ (propagation transverse to the initial stress), then $D = 0$ and again $\phi = 0$ or $\pi/2$, transverse and longitudinal, respectively, with wave speeds given by $\rho v_{\mathrm{T}}^2 = A + B$ and $\rho v_{\mathrm{L}}^2 = A + C$. There is, however, an additional case not covered previously in which a longitudinal and transverse wave can propagate. By setting $\phi = \theta$ in (A.9) and discarding cases already discussed we obtain $B\cos\theta + D = 0$, which leads to (on discarding a factor $\tau \neq 0$)

$$2\beta_1 + \beta_2 + (2\gamma_1 + \gamma_2)\tau + (\beta_3 + 2\gamma_3\tau + \gamma_4\tau^2)\tau\cos^2\theta = 0. \tag{A.10}$$

If this has a solution (or solutions) θ for $\cos^2\theta < 1$, then a longitudinal wave can propagate in the direction defined by such an angle (or angles). The corresponding wave speed is given by

$$\rho v_{\mathrm{L}}^2 = A + C + D\cos\theta = 2\alpha_1 + \alpha_2 + [1 + 2\beta_1 + \beta_2 + (2\gamma_1 + \gamma_2)\tau)]\tau\cos^2\theta. \tag{A.11}$$

Equally, by setting $\phi = \theta + \pi/2$, the same reduction $B\cos\theta + D = 0$ is obtained and a transverse wave can accompany the longitudinal wave and has wave speed which, on use of (A.10), can be written

$$\rho v_{\mathrm{T}}^2 = A + B + D\cos\theta = \alpha_1 + \beta_1\tau + \gamma_1\tau^2 + \tau\cos^2\theta - [2\beta_1 + \beta_2 + (2\gamma_1 + \gamma_2)\tau]\tau\sin^2\theta. \tag{A.12}$$

For the specialization (6.32), (A.10) is satisfied and (A.11) and (A.12) reduce to $\rho v_{\mathrm{L}}^2 = \lambda_0 + 2\mu_0 + \tau\cos^2\theta$ and $\rho v_{\mathrm{T}}^2 = \mu_0 - \frac{1}{2}\tau + \tau\cos^2\theta$, respectively. Formulas in Section 6.4 are recovered by taking $\theta = 0$ and $\theta = \frac{1}{2}\pi$.

With reference to (A.6), we now consider the special cases corresponding to vanishing of one or other of the coefficients of **a** and **n**. These are $B(\mathbf{m}\cdot\mathbf{a}) + D(\mathbf{m}\cdot\mathbf{n}) = 0$ with $C(\mathbf{m}\cdot\mathbf{n}) + D(\mathbf{m}\cdot\mathbf{a}) \neq 0$ and $B(\mathbf{m}\cdot\mathbf{a}) + D(\mathbf{m}\cdot\mathbf{n}) \neq 0$ with $C(\mathbf{m}\cdot\mathbf{n}) + D(\mathbf{m}\cdot\mathbf{a}) = 0$. The first of these corresponds to the case $B\cos\theta + D = 0$ just considered, while for the second, **m** is aligned with the direction of uniaxial initial stress and $C\cos\theta + D = 0$, which yields the nontrivial result

$$\alpha_1 + \alpha_2 + [\beta_1 + \beta_2 + (\gamma_1 + \gamma_2)\tau]\tau = 0. \tag{A.13}$$

This puts no restriction on the direction of propagation **n** and the wave speed is given by

$$\rho v^2 = A + B + D\cos\theta \tag{A.14}$$

as in (A.12), but the wave is not necessarily transverse in this case. For the special case (6.32), the condition (A.13) yields $\tau = -2(\lambda_0 + \mu_0)$, as for (A.3), and ρv^2 specializes to $\mu_0 + (\lambda_0 + \mu_0)\sin^2\theta$.