Surface stability of nonlinear magnetoelastic solids

M. Ottenio[*1], **M. Destrade**[2], and **R.W. Ogden** [3]

[1] Université Lyon 1, Lyon, 69000; IUTB Bâtiment Central; 17, rue de France; 69627 Villeurbanne Cedex; France.

[2] Institut Jean le Rond d'Alembert; CNRS (UMR7190); Université Pierre et Marie Curie; Case 162, 4 Place Jussieu; 75282 Paris Cedex 05; France.

[3] Department of Mathematics; University of Glasgow; University Gardens; Glasgow, G12 8QW; Scotland, UK.

The present paper proposes to identify surface stability when a magnetoelastic half-space is subjected to a pure homogeneous pre-deformation and to a magnetic field normal to its (plane) boundary. Clearly, the aim is to find the critical stretch ratio beyond which surface instabilities may develop, or in other words, to establish a bifurcation criterion based on the incremental static solution of the boundary value problem. We want to analyse how the presence of a coupling between magnetism and nonlinear elasticity modify the conditions of stability.

1 Introduction

Magnetoelastic materials are smart because they can respond to a mechanical solicitation, but also to a magnetic solicitation. This multiphysical ability allow engineers to imagine always smarter industrial applications such as vibration absorbers, whose mechanical damping is tuned by applying suitable magnetic fields, or smart biosensors to measure body fluid acidity, for instance. These technical advances naturally need to be accompanied by a strong theoretical modelling (see [1], [2], [3], or more recently [4], [5], [6]). Here, we propose a theoretical study of the surface stability of nonlinear magnetoelastic solids, based on incremental static solutions of the boundary value problem as Biot [7] successfully did in nonlinear elasticity. More details about the referring incremental theory can be found in [8].

2 Pure homogeneous deformation of a magnetoelastic half-space

Let X_1, X_2, X_3 be the rectangular Cartesian coordinates attached to a configuration $\mathcal{B}_0$ of an incompressible, isotropic, magnetoelastic half-space of boundary $X_2 = 0$, and occupying $X_2 > 0$. After a plane strain in the plane (X_1, X_2), the particles of the half-space located at $\boldsymbol{X}$ in $\mathcal{B}_0$ occupy the positions $x_1 = \lambda X_1$, $x_2 = \lambda^{-1}X_2$, and $x_3 = X_3$ in the new configuration $\mathcal{B}$. We refer to λ as the principal stretch in the X_1 direction. Then, the components of the deformation gradient $\boldsymbol{F}$ are $F_{ij} = \partial x_i / \partial X_j$. We choose the magnetic induction vector $\boldsymbol{B}$, defined in $\mathcal{B}$, to be expressed as $\boldsymbol{B} = [0, B_2, 0]^{\mathsf{T}}$. Constitutive laws for magnetoelastomers are now specified by adopting Dorfmann and Ogden's formulation [4] which involves a modified free energy function per unit volume, denoted by Ω, such that

$$\boldsymbol{T} = \frac{\partial \Omega(\boldsymbol{F}, \boldsymbol{B}_l)}{\partial \boldsymbol{F}} - p\boldsymbol{F}^{-1}, \qquad \boldsymbol{H}_l = \frac{\partial \Omega}{\partial \boldsymbol{B}_l}, \tag{1}$$

where p is a Lagrange multiplier associated with the incompressibility constraint. The tensor $\boldsymbol{T}$ represents the Lagrangian version of the total Cauchy stress tensor $\boldsymbol{\tau}$ in $\mathcal{B}$. This tensor $\boldsymbol{\tau}$ includes the Cauchy and Maxwell stress tensors and is introduced so that the equilibrium equations in the absence of mechanical body force is $\operatorname{div} \boldsymbol{\tau} = \boldsymbol{0}$. The tensors $\boldsymbol{B}_l$ and $\boldsymbol{H}_l$ represent the Lagrangian versions of $\boldsymbol{B}$ and of the magnetic vector $\boldsymbol{H}$ in $\mathcal{B}$, respectively.

## 3 Incremental static boundary value problem

We now consider that both magnetic fields and the deformation within the material undergo incremental changes. The increment $\dot{\boldsymbol{F}}$ in $\boldsymbol{F}$ is then related to an incremental displacement vector $\boldsymbol{u}$ through $\boldsymbol{F} = (\partial \boldsymbol{u}/\partial \boldsymbol{x})\boldsymbol{F} = \boldsymbol{dF}$, where $\boldsymbol{d}$ has just been introduced. Following small-on-large theory, the constitutive laws (1) are first incremented (superposed dot) before being transformed into their Eulerian counterparts in $\mathcal{B}$ (indicated by a 0 subscript) giving

$$\dot{\boldsymbol{T}}_0 = \boldsymbol{\mathcal{A}}_0 \boldsymbol{d} + \boldsymbol{\Gamma}_0 \dot{\boldsymbol{B}}_{l_0} + p\boldsymbol{d} - \dot{p}\boldsymbol{I}, \qquad \dot{\boldsymbol{H}}_{l_0} = \boldsymbol{\Gamma}_0 \boldsymbol{d} + \boldsymbol{\mathcal{K}}_0 \dot{\boldsymbol{B}}_{l_0}, \tag{2}$$

where $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\Gamma}_0$, and $\boldsymbol{\mathcal{K}}_0$ represent the instantaneous magnetoelastic moduli tensors. In index notation, they are defined by

$$\mathcal{A}_{0jisk} = F_{j\alpha}F_{s\beta}\frac{\partial^2\Omega}{\partial F_{i\alpha}\partial F_{k\beta}}, \quad \Gamma_{0jik} = F_{j\alpha}F_{\beta k}^{-1}\frac{\partial^2\Omega}{\partial F_{i\alpha}\partial B_{l_\beta}}, \quad \mathcal{K}_{0ij} = F_{\alpha i}^{-1}F_{\beta j}^{-1}\frac{\partial^2\Omega}{\partial B_{l_\alpha}\partial B_{l_\beta}}. \tag{3}$$

* Corresponding author: e-mail: melanie.ottenio@inrets.fr, Phone: +33 (0)4 72 14 23 85

(a) $\alpha = 0.5$ (b) $\alpha = 2.0$
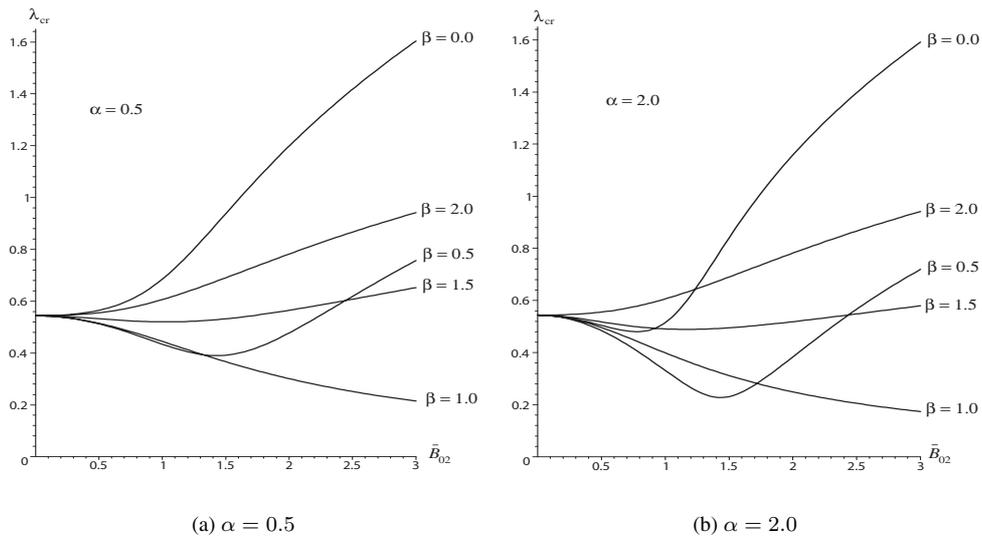
Fig. 1 Dependence of the critical stretch $\lambda_{\mathrm{cr}} < 1$ for instability in compression for a magnetoelastic Mooney-Rivlin solid in plane strain on the non-dimensional measure $\bar{B}_{l2}$ of the magnetic field for several values of the magnetoelastic coupling parameters α and β.

All the equations describing the finite magnetoelastic problem have to be incremented in the same manner. It means that incremental versions of the mechanical equilibrium, of Maxwell's equations (inside and outside the material), and of the jump conditions for the total stress and the magnetic fields are required. Then, finding surface stabilities turns out to seek small-amplitude solutions, localized near the interface $x_2 = 0$. In other words, solutions can take the form $A e^{-ksx_2} e^{ikx_1}$, where A is an arbitrary parameter, k is in relation with the wavelength of the perturbation $(= 2\pi/k)$, and s is an unknown parameter such that the condition $\Re(s) > 0$ is checked to ensure decay with increasing x_2. Using this solution form, the full set of incremental equations allow us to obtain a 7×7 matrix, whose vanishing of the determinant provides the bifurcation criterion.

4   Surface stability of a Mooney-Rivlin magneto-elastic solid

We propose here to specialize the modified free energy Ω so that

$$\Omega = \frac{1}{4}\mu(0)[(1+\gamma)(\lambda^2 + \lambda^{-2} - 2) + (1-\gamma)(\lambda^2 + \lambda^{-2} - 2)] + \mu_0^{-1}(\alpha B_{l_2}^2 + \lambda^{-2}\beta B_{l_2}^2), \qquad (4)$$

where $\mu(0)$ is the shear modulus of the material without magnetic fields; μ_0 is the magnetic permeability in vacuum; γ is a dimensionless elastic material constant, and α and β are dimensionless magnetoelastic coupling parameters. If α and β vanish, the material is a classical Mooney-Rivlin material. Two parameters (α and β) are at least necessary to capture the two-way coupling in the sense than stress can be affected by the magnetic field, but magnetic properties can also be changed by the deformation. Then, the static solutions of the incremental boundary value problem of Section 3 are the critical stretch ratios λ_{cr} in compression, which are expressed as a function of $\bar{B}_{l_2} = B_{l_2}/\sqrt{\mu_0\mu(0)}$ (see Figure 1 for graphical results). At $\bar{B} = 0$, the well-known critical stretch ratio for elastic Mooney-Rivlin materials in plane-strain is recovered ($\lambda_{cr} = 0.5437$). When lower values of λ_{cr} are reached, the magnetic field has a stabilizing effect on the magnetoelastic half-space. Otherwise, the magnetoelastic half-space is unstable in compression, and even in tension (when $\lambda_{cr} > 1$).

References

[1] C. Truesdell and R. Toupin, The classical field theories, Handbuch der Physik, volume III/1 (Springer, Berlin, 1960).
[2] W. Brown, Magnetoelastic Interactions (Springer, Berlin, 1966).
[3] A. Eringen and G. Maugin, Electrodynamics of Continua (Springer, New York, 1990).
[4] A. Dorfmann and R. Ogden, Q. J. Mech. Appl. Math. **57**, 599–622 (2004).
[5] A. Dorfmann and I. Brigadnov, Comp. Mater. Sci. **29**, 270–282 (2004).
[6] S. Kankanala and N. Triantafyllidis, J. Mech. Phys. Solids **52**, 2869–2908 (2004).
[7] M. Biot, Mechanics of Incremental Deformations (John Wiley, New York, 1965).
[8] M. Ottenio, M. Destrade, and R. Ogden, J Elasticity (under press).