# Linear Elastodynamics and Waves

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## Contents

**Abstract**

We provide a simple introduction to wave propagation in the framework of linear elastodynamics. We discuss bulk waves in isotropic and anisotropic linear elastic materials and we survey several families of surface and interface waves. We conclude by suggesting a list of books for a more detailed study of the topic.

**Keywords**: linear elastodynamics, anisotropy, plane homogeneous waves, bulk waves, interface waves, surface waves.

# 1 Introduction

In elastostatics we study the equilibria of elastic solids; when its equilibrium is disturbed, a solid is set into *motion*, which constitutes the subject of elastodynamics. Early efforts in the study of elastodynamics were mainly aimed at modeling seismic wave propagation. With the advent of electronics, many applications have been found in the industrial world. These include the manufacturing of high frequency acoustic wave filters and transducers (used in everyday electronic device such as global positioning systems, cell phones, miniature motors, detectors, sensors, etc.), the health monitoring of of elastic structures (non-destructive evaluation in the automotive or aeronautic industry), the acoustic determination of elastic properties of solids (physics, medicine, engineering, etc.), ultrasonic imaging techniques (medicine, oil prospection, etc.), and so on.

The local excitation of a body is not instantaneously detected at a distance away from the source of excitation. It takes time for a disturbance to propagate from one point to another, which is why elastodynamics relies heavily on the study of *waves*. Everyone is familiar with the notion of wave, but the broad use of this term makes it difficult to produce a precise definition. For this reason we shall consider the notion of wave as primitive.

A fundamental mathematical representation of a wave is

$$u(x,t) = f(x - vt) \tag{1.1}$$

where $f$ is a function of the variable $\xi = x - vt$ and $v$ is a nonzero constant. Waves represented by functions of the form (1.1) are called *traveling waves*. For such waves the initial profile $u(x,0) = f(x)$ is translated along the $x-$axis at a speed $|v|$. For this reason traveling waves are also called *waves of permanent profile* or *progressive plane waves*.

Traveling waves are a most important class of functions because the general solution of the classical one-dimensional wave equation: $u_{tt} = v^2 u_{xx}$ is the sum of two of such waves, one, $F(x-vt)$, moving right with speed $v$, and the other, $G(x+vt)$ moving left, also with speed $v$:

$$u(x,t) = F(x-vt) + G(x+vt), \tag{1.2}$$

where $F$ and $G$ are arbitrary functions. Therefore the solution to any initial value problem in the entire real line $-\infty < x < \infty$ of the wave equation can be written in terms of such two traveling waves via the well known d'Alembert form.

When the functions $F$ and $G$ in (1.2) are sines or cosines we speak of *plane harmonic waves*. This is the case if

$$f = A\cos k(x-vt), \tag{1.3}$$

where $A$, $k$, $v$ are constant scalars. The motion (1.3) describes a wave propagating with *amplitude* $A$, *phase speed* $v$, *wavenumber* $k$, wavelength $2\pi/k$, angular *frequency* $\omega = kv$, and temporal period $2\pi/\omega$. For mathematical convenience, the wave (1.2) can be represented as

$$f = \{A\exp \mathrm{i}k(x-vt)\}^+, \tag{1.4}$$

where $\{\bullet\}^+$ designates the real part of the complex quantity.

In a three-dimensional setting, a plane harmonic wave propagating in the direction of the unit vector $\boldsymbol{n}$ is described by the mechanical displacement vector,

$$\boldsymbol{u} = \{\boldsymbol{A}\exp \mathrm{i}k(\boldsymbol{n}\cdot\boldsymbol{x} - vt)\}^+, \tag{1.5}$$

where $\boldsymbol{A}$ is the amplitude vector, possibly complex. If $\boldsymbol{A}$ is the multiple of a real vector: $\boldsymbol{A} = \alpha\boldsymbol{a}$ say, where $\alpha$ is a scalar and $\boldsymbol{a}$ a real unit vector, then the wave is *linearly polarized*; in particular when $\boldsymbol{a}\times\boldsymbol{n} = \boldsymbol{0}$, the wave is a linearly polarized *longitudinal wave*, and when $\boldsymbol{a}\cdot\boldsymbol{n} = 0$, the wave is a linearly polarized *transverse wave*. Otherwise (when $\boldsymbol{A} \neq \alpha\boldsymbol{a}$) the wave is *elliptically polarized*; in particular when $\boldsymbol{A}.\boldsymbol{n} = 0$, the wave is an elliptically polarized transverse wave.

A three-dimensional setting allows for the description of more complex wave phenomena; for example it is possible to investigate the following interesting generalization of the solutions in (1.5),

$$\boldsymbol{u} = \{g(\boldsymbol{m}\cdot\boldsymbol{x})\boldsymbol{A}\exp \mathrm{i}k(\boldsymbol{n}\cdot\boldsymbol{x} - vt)\}^+, \tag{1.6}$$

where $\boldsymbol{m}$ is another unit vector and $g$ is the amplitude function. The planes $\boldsymbol{n}\cdot\boldsymbol{x} =$ constant are the *planes of constant phase*, and the planes $\boldsymbol{m}\cdot\boldsymbol{x} =$ constant are the *planes of constant amplitude*. When $\boldsymbol{n}$ and $\boldsymbol{m}$ are

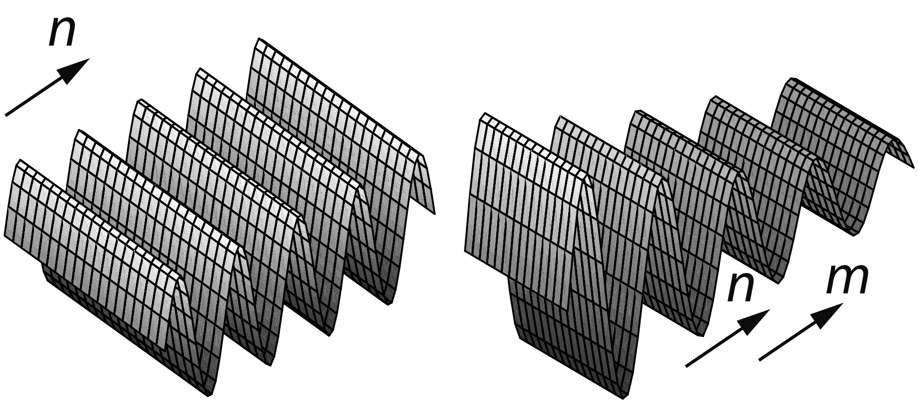


Figure 1: Homogeneous waves. On the left: a wave with constant amplitude; On the right: a wave with an attenuated amplitude.

parallel, the waves are said to be *homogeneous*; this class includes (1.5) as a special case, but also waves for which the amplitude varies in the direction of propagation, such as attenuated homogeneous waves, see Figures 1. When $\boldsymbol{n} \times \boldsymbol{m} \neq \boldsymbol{0}$, the waves are *inhomogeneous*; this class includes a wave which propagates harmonically in one direction while its amplitude decays in another, see Figures 2.

In the case of homogeneous waves (1.5), note that instead of the *wave vector/speed* couple $(\boldsymbol{k}, v)$ where $\boldsymbol{k} = k\boldsymbol{n}$, the *slowness vector/frequency* couple $(\boldsymbol{s}, \omega)$ where $\boldsymbol{s} = v^{-1}\boldsymbol{n}$, can be used equivalently, giving the representation:

$$\boldsymbol{u} = \{\boldsymbol{A} \exp \mathrm{i}\omega(\boldsymbol{s} \cdot \boldsymbol{x} - t)\}^+. \tag{1.7}$$

Similarly for inhomogeneous plane waves, a complex slowness vector can be introduced: $\boldsymbol{S} = \boldsymbol{S}^+ + \mathrm{i}\boldsymbol{S}^-$, giving the waves:

$$\boldsymbol{u} = \{\boldsymbol{A} \exp \mathrm{i}\omega(\boldsymbol{S} \cdot \boldsymbol{x} - t)\}^+, \tag{1.8}$$

as a sub-case of (1.6). Here $\omega$ is the real angular frequency, the planes of constant phase are $\boldsymbol{S}^+ \cdot \boldsymbol{x}$ =constant and the planes of constant amplitude are $\boldsymbol{S}^- \cdot \boldsymbol{x}$ =constant. The phase speed is $v = 1/\left|\boldsymbol{S}^+\right|$ and the attenuation factor is $\left|\boldsymbol{S}^-\right|$.

As a train of waves propagates, it carries energy. It can be shown that for time-harmonic waves, the ratio of the mean energy flux to the mean energy density, $\boldsymbol{v}_g$ say, is computed as

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}, \qquad [\boldsymbol{v}_g]_i = \frac{\partial \omega}{\partial k_i}. \tag{1.9}$$

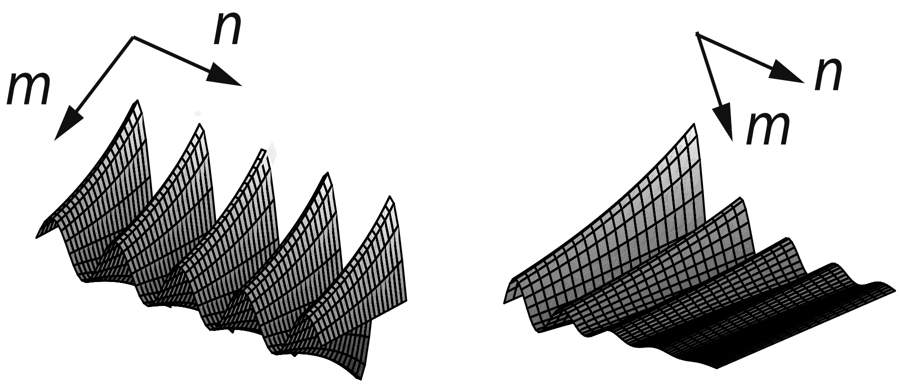


Figure 2: Inhomogeneous waves. On the left: the planes of constant phase are orthogonal to the planes of constant amplitude; On the right: they are at 45°.

This vector is called the *group velocity*.

When we consider the wave equation in a semi-infinite domain or in a finite domain, the resolution can become quite complex, because we now have to satisfy not only initial conditions but also *boundary conditions*. In this chapter we study some wave solutions to the equations of elastodynamics in the case of an infinite medium, and then for some simple boundary conditions, with a view to demonstrate the usefulness and versatility of homogeneous and inhomogeneous plane waves.

# 2 Bulk waves

We start by considering the propagation of waves in an infinite elastic medium. They are often called *bulk waves*, because they travel within the bulk of a solid with dimensions which are large compared to the wavelength, so that boundary effects can be ignored (think for instance of the waves triggered deep inside the Earth crust by a seismic event.)

Referred to a rectangular Cartesian coordinate system $(Ox_1x_2x_3)$, say, the particle displacement components are denoted $(u_1, u_2, u_3)$, and the strains are given by

$$2\epsilon_{ij} \equiv u_{i,j} + u_{j,i}, \tag{2.1}$$

where the comma denotes partial differentiation with respect to the Cartesian coordinates $x_j$.

The constitutive equations for a general anisotropic homogeneous elastic material are

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}, \tag{2.2}$$

where the elastic stiffnesses $c_{ijkl}$ are constants, with the symmetries

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{2.3}$$

Because of these symmetries there are at most twenty-one independent elastic constants.

The equations of motion, in the absence of body forces, read

$$\text{div}\ \boldsymbol{\sigma} = \rho\partial^2\boldsymbol{u}/\partial t^2, \tag{2.4}$$

where $\rho$ is the mass density.

Inserting the constitutive equation (2.2) into the equations of motions (2.4), we obtain

$$c_{ijkl}u_{k,lj} = \rho\partial^2 u_i/\partial t^2. \tag{2.5}$$

For *isotropic* elastic materials, 12 elastic stiffnesses are zero, and the remaining 9 are given in terms of 2 independent material constants: $\lambda$ and $\mu$, the so-called *Lamé coefficients*. In that case the $c_{ijkl}$ can be written as

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.6}$$

Then the constitutive equations (2.2) reduce to

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{2.7}$$

and the equations of motions (2.5) read

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} = \rho\partial^2 u_i/\partial t^2. \tag{2.8}$$

Writing down the governing equations for *anisotropic* elastic materials is a more complex operation than what we have just done for isotropic materials.

For example let us consider *transversely isotropic materials*. They have 12 elastic stiffnesses which are zero, whilst the remaining 9 can be expressed in terms of 5 independent material constants. When the axis of symmetry is along $x_3$, the constitutive equations (2.2) read

$$\begin{aligned}
\sigma_{11} &= d_{11}u_{1,1} + d_{12}u_{2,2} + d_{33}u_{3,3}, \\
\sigma_{22} &= d_{12}u_{1,1} + d_{11}u_{2,2} + d_{13}u_{3,3}, \\
\sigma_{33} &= d_{13}(u_{1,1} + u_{2,2}) + d_{33}u_{3,3}, \\
\sigma_{13} &= d_{44}(u_{1,3} + u_{3,1}), \\
\sigma_{23} &= d_{44}(u_{2,3} + u_{3,2}), \\
\sigma_{12} &= \tfrac{1}{2}(d_{11} - d_{12})(u_{1,2} + u_{2,1}).
\end{aligned} \tag{2.9}$$

where $d_{11}$,$d_{12}$, $d_{13}$, $d_{33}$, $d_{44}$ are independent material parameters. The equations of motion (2.4) now read

$$\begin{aligned}
&d_{11}u_{1,11} + \tfrac{1}{2}(d_{11} - d_{12})u_{1,22} + d_{44}u_{1,33} \\
&\qquad + \tfrac{1}{2}(d_{11} + d_{12})u_{2,12} + (d_{13} + d_{44})u_{3,13} = \rho\partial^2 u_1/\partial t^2, \\
&d_{11}u_{2,11} + \tfrac{1}{2}(d_{11} + d_{12})u_{1,12} \\
&\qquad + d_{44}u_{2,33} + \tfrac{1}{2}(d_{11} - d_{12})u_{2,11} + (d_{13} + d_{44})u_{3,23} = \rho\partial^2 u_2/\partial t^2, \\
&(d_{13} + d_{44})(u_{1,13} + u_{2,23}) + d_{44}(u_{3,11} + u_{3,22}) + d_{33}u_{3,33} = \rho\partial^2 u_3/\partial t^2. \quad (2.10)
\end{aligned}$$

As another example, consider *cubic materials*. They also have 12 elastic stiffnesses which are zero, whilst the remaining 9 can be expressed in terms of only 3 material constants. Here the constitutive equations (2.2) read

$$\begin{aligned}
\sigma_{11} &= d_{11}u_{1,1} + d_{12}(u_{2,2} + u_{3,3}), \\
\sigma_{22} &= d_{11}u_{2,2} + d_{12}(u_{1,1} + u_{3,3}), \\
\sigma_{33} &= d_{11}(u_{3,3} + d_{12}(u_{1,1} + u_{2,2}), \\
\sigma_{13} &= d_{44}(u_{1,3} + u_{3,1}), \\
\sigma_{23} &= d_{44}(u_{2,3} + u_{3,2}), \\
\sigma_{12} &= d_{44}(u_{1,2} + u_{2,1}), \qquad (2.11)
\end{aligned}$$

where $d_{11}$, $d_{12}$, and $d_{44}$ are independent material parameters. The corresponding equations of motion read

$$\begin{aligned}
&d_{11}u_{1,11} + d_{44}(u_{1,22} + u_{1,33}) + (d_{12} + d_{44})(u_{2,12} + u_{3,13}) = \rho\partial^2 u_1/\partial t^2, \\
&d_{11}u_{2,22} + d_{44}(u_{2,11} + u_{2,33}) + (d_{12} + d_{44})(u_{1,12} + u_{3,23}) = \rho\partial^2 u_2/\partial t^2, \\
&d_{11}u_{3,33} + d_{44}(u_{3,11} + u_{3,22}) + (d_{12} + d_{44})(u_{1,13} + u_{2,23}) = \rho\partial^2 u_3/\partial t^2. \quad (2.12)
\end{aligned}$$

## 2.1 Homogeneous waves in isotropic solids

Consider the propagation of homogeneous plane waves of constant amplitude in a homogeneous isotropic elastic material. Therefore, search for solutions in the form (1.5) to the equations of motion (2.8) .

Introducing (1.5) into (2.8), gives the *propagation condition*

$$\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{A} = \rho v^2 \boldsymbol{A}, \qquad (2.13)$$

where the *acoustical tensor* $\boldsymbol{Q}(\boldsymbol{n})$ is given by

$$\boldsymbol{Q}(\boldsymbol{n}) = (\lambda + \mu)\boldsymbol{n} \otimes \boldsymbol{n} + \mu\boldsymbol{I}, \qquad (2.14)$$

Introducing any two unit vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ forming an orthonormal triad with $\boldsymbol{n}$ and using the decomposition $\boldsymbol{I} = \boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{m} \otimes \boldsymbol{m} + \boldsymbol{p} \otimes \boldsymbol{p}$, we write

$$\boldsymbol{Q}(\boldsymbol{n}) = (\lambda + 2\mu)\boldsymbol{n} \otimes \boldsymbol{n} + \mu(\boldsymbol{m} \otimes \boldsymbol{m} + \boldsymbol{p} \otimes \boldsymbol{p}), \tag{2.15}$$

Non-trivial solutions to the algebraic eigenvalue problem (2.13) exist only if the following *secular equation*

$$\det\big(\boldsymbol{Q}(\boldsymbol{n}) - \rho v^2 \boldsymbol{I}\big) = 0, \tag{2.16}$$

is satisfied. Here it factorizes into

$$(\lambda + 2\mu - \rho v^2)(\mu - \rho v^2)^2 = 0, \tag{2.17}$$

giving a simple eigenvalue and a double eigenvalue. Observing (2.15), we find that $\boldsymbol{A} = \boldsymbol{n}$ is an eigenvector of $\boldsymbol{Q}(\boldsymbol{n})$, associated with the simple eigenvalue

$$\rho v_L^2 = \lambda + 2\mu. \tag{2.18}$$

It corresponds to a linearly polarized homogeneous longitudinal bulk wave, traveling with speed $v_L$. We also find that $\boldsymbol{A} = \boldsymbol{m} + \alpha\boldsymbol{p}$, where $\alpha$ is an arbitrary scalar, is an eigenvector of $\boldsymbol{Q}(\boldsymbol{n})$ with the double eigenvalue

$$\rho v_T^2 = \mu. \tag{2.19}$$

It corresponds to an elliptically polarized homogeneous transverse wave, traveling with speed $v_T$. Sub-cases of polarization types include linear polarization when $\alpha$ is real, and circular polarization when $\alpha = \pm\mathrm{i}$.

Note that the speeds do not depend on the direction of propagation $\boldsymbol{n}$, as expected in the case of isotropy. Note also that the wave speeds are real when

$$\lambda + 2\mu > 0, \qquad \mu > 0. \tag{2.20}$$

Table 1 report longitudinal and transverse bulk wave speeds as computed for several isotropic solids.

Table 1. *Material parameters of 5 different linear isotropic elastic solids: mass density* ($10^3$ kg/m$^3$) *and Lamé coefficients* ($10^{10}$ N/m$^2$); *Computed speeds of the shear and of the longitudinal homogeneous bulk waves* (m/s).

| Material | $\rho$ | $\lambda$ | $\mu$ | $v_T$ | $v_L$ |
|---|---|---|---|---|---|
| Silica | 2.2 | 1.6 | 3.1 | 3754 | 5954 |
| Aluminum | 2.7 | 6.4 | 2.5 | 3043 | 6498 |
| Iron | 7.7 | 11.0 | 7.9 | 3203 | 5900 |
| Steel | 7.8 | 8.6 | 7.9 | 3182 | 5593 |
| Nickel | 8.9 | 20.0 | 7.6 | 2922 | 6289 |

## 2.2 Homogeneous waves in anisotropic solids

Now search for solutions in the form (1.5) to the equations of motion in anisotropic solids (2.5). Then the eigenvalue problem (2.13) follows, now with an anisotropic acoustic tensor $\boldsymbol{Q}(\boldsymbol{n})$ with components depending on the propagation direction $\boldsymbol{n}$:

$$Q_{ik}(\boldsymbol{n}) = c_{ijkl}n_l n_j = Q_{ki}(\boldsymbol{n}). \tag{2.21}$$

In general the secular equation (2.16) is a cubic in $v^2$. It can be shown that when the roots are simple, the corresponding eigenvectors are proportional to three real vectors, mutually orthogonal. Hence in general there are three linearly polarized waves for a given propagation direction. If for some particular $\boldsymbol{n}$, the secular equation has a double or triple root, then a circularly-polarized wave may propagate in that direction, see the isotropic case for an example.

Consider a *transversally isotropic* solid: there the components of the acoustic tensor are found from the equations of motion (2.10) as

$$\begin{aligned}
&Q_{11}(\boldsymbol{n}) = d_{11}n_1^2 + \tfrac{1}{2}(d_{11} - d_{12})n_2^2 + d_{44}n_3^2, && Q_{12}(\boldsymbol{n}) = \tfrac{1}{2}(d_{11} + d_{12})n_1 n_2,\\
&Q_{22}(\boldsymbol{n}) = \tfrac{1}{2}(d_{11} - d_{12})n_1^2 + d_{11}n_2^2 + d_{44}n_3^2, && Q_{23}(\boldsymbol{n} = (d_{13} + d_{44})n_2 n_3,\\
&Q_{33}(\boldsymbol{n}) = d_{22}(n_1^2 + n_2^2) + d_{33}n_3^2, && Q_{13}(\boldsymbol{n}) = (d_{13} + d_{44})n_1 n_3,
\end{aligned} \tag{2.22}$$

Here the secular equation (2.16) factorizes into the product of a term linear in $\rho v^2$ and a term quadratic in $\rho v^2$. The linear term gives the eigenvalue

$$\rho v_2^2 = \tfrac{1}{2}(d_{11} - d_{12})(n_1^2 + n_2^2) + d_{44}n_3^2, \tag{2.23}$$

and it can be checked that the associated eigenvector is $\boldsymbol{A} = [n_2, -n_1, 0]^T$. It corresponds to a linearly polarized transverse wave, traveling with speed $v_2$. The quadratic is too long to reproduce here; in general it yields two linearly polarized waves which are neither purely longitudinal nor transverse, except in certain special circumstances, of which a few examples are presented below.

If the wave propagates along the $x_1$ axis, then $\boldsymbol{n} = [1, 0, 0]^T$ and the secular equation factorizes into $(d_{11} - \rho v^2)(\frac{1}{2}(d_{11} - d_{12}) - \rho v^2)(d_{44} - \rho v^2) = 0$, giving the three eigenvalues $\rho v_1^2 = d_{11}$, $\rho v_2^2 = \frac{1}{2}(d_{11} - d_{12})$ (in accordance with (2.23)), and $\rho v_3^2 = d_{44}$. The eigenvector corresponding to $\rho v_1^2$ is $\boldsymbol{A} = \boldsymbol{e}_1$, the unit vector along $x_1$, giving the wave

$$\boldsymbol{u} = \{\exp \mathrm{i}k(x_1 - v_1 t)\}^+ \boldsymbol{e}_1, \tag{2.24}$$

a linearly polarized longitudinal wave. The eigenvectors corresponding to $\rho v_2^2$ and $\rho v_3^2$ are $\boldsymbol{A} = \boldsymbol{e}_2$ and $\boldsymbol{A} = \boldsymbol{e}_3$, respectively, the unit vectors along $x_2$ and $x_3$, giving the two waves:

$$\boldsymbol{u} = \{\exp \mathrm{i}k(x_1 - v_2 t)\}^+ \boldsymbol{e}_2, \qquad \boldsymbol{u} = \{\exp \mathrm{i}k(x_1 - v_3 t)\}^+ \boldsymbol{e}_3, \tag{2.25}$$

two linearly polarized transverse waves.

If the wave propagates along the $x_3$ axis, then $\boldsymbol{n} = [0,0,1]^T$, and the secular equation factorizes into $(d_{33} - \rho v^2)(d_{44} - \rho v^2)^2 = 0$ giving a simple eigenvalue $\rho v_1^2 = d_{33}$ and a double eigenvalue $\rho v_2^2 = d_{44}$. The corresponding solutions are a linearly polarized longitudinal wave:

$$\boldsymbol{u} = \{\exp \mathrm{i}k(x_3 - v_1 t)\}^+ \boldsymbol{e}_3, \tag{2.26}$$

and an elliptically polarized transverse wave

$$\boldsymbol{u} = \{\exp \mathrm{i}k(x_3 - v_2 t)\}^+ (\boldsymbol{e}_1 + \alpha \boldsymbol{e}_2), \tag{2.27}$$

where $\alpha$ is an arbitrary scalar. Note that a circularly polarized wave is possible for $\alpha = \pm\mathrm{i}$, as expected when the secular equation has a double root.

Directions along which circularly polarized waves exist are called the *acoustic axes*. To determine whether there are acoustic axes in a given anisotropic solid is equivalent to finding whether the secular equation admits a double root. In the present case, this could happen (a) if the determinant of the quadratic term in the secular equation is zero, or (b) if the eigenvalue (2.23) is also a root of the quadratic term. It can be shown that (a) is never possible, whereas (b) is always possible.

Consider a *cubic* solid: there the acoustic tensor is found from the equations of motion (2.12) as

$$\boldsymbol{Q}(\boldsymbol{n}) = \begin{bmatrix} (d_{11} - d_{44})n_1^2 + d_{44} & (d_{12} + d_{44})n_1 n_2 & (d_{12} + d_{44})n_1 n_3 \\ (d_{12} + d_{44})n_1 n_2 & (d_{11} - d_{44})n_2^2 + d_{44} & (d_{12} + d_{44})n_2 n_3 \\ (d_{12} + d_{44})n_1 n_3 & (d_{12} + d_{44})n_2 n_3 & (d_{11} - d_{44})n_3^2 + d_{44} \end{bmatrix}. \tag{2.28}$$

Here all three axes $x_1$, $x_2$, and $x_3$ are equivalent. For propagation along $x_1$ for instance, $\boldsymbol{n} = [1,0,0]^T$ and the acoustical tensor is diagonal. There is one linearly polarized longitudinal wave traveling at speed $\sqrt{d_{11}/\rho}$, and an elliptically (and thus possibly, circularly) polarized transverse wave traveling at speed $\sqrt{d_{44}/\rho}$. Similarly when $\boldsymbol{n} = [0,1,0]^T$ and when $\boldsymbol{n} = [0,0,1]^T$. In each case, $\rho v_2^2 = d_{44}$ is a double eigenvalue, showing that the symmetry axes are acoustic axes.

Now take $\boldsymbol{n} = [\pm 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]^T$, or $\boldsymbol{n} = [1/\sqrt{3}, \pm 1/\sqrt{3}, 1/\sqrt{3}]^T$, or $\boldsymbol{n} = [1/\sqrt{3}, 1/\sqrt{3}, \pm 1/\sqrt{3}]^T$. These directions are acoustic axes, because then

$$\boldsymbol{Q}(\boldsymbol{n}) = \tfrac{1}{3}(d_{11} - d_{12} + d_{44})\boldsymbol{I} + (d_{12} + d_{44})\boldsymbol{n} \otimes \boldsymbol{n}, \tag{2.29}$$

clearly showing the existence of a linearly polarized longitudinal wave (any $\boldsymbol{A}$ such that $\boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{0}$) and of an elliptically polarized transverse wave (any $\boldsymbol{A}$ such that $\boldsymbol{A} \cdot \boldsymbol{n} = 0$), which can be circularly polarized. They travel with speeds $v_1$ and $v_2$, respectively, given by

$$\rho v_1^2 = \tfrac{1}{3}(d_{11} + 2d_{12} + 4d_{44}), \qquad \rho v_2^2 = \tfrac{1}{3}(d_{11} - d_{12} + d_{44}), \tag{2.30}$$

the first eigenvalue being simple and the second, double.

## 2.3 Slowness surface and wavefronts

In anisotropic solids, the three phase speeds obtained by solving the secular equation for $v$ depend on the propagation direction $\boldsymbol{n}$. Let us call them $v_1(\boldsymbol{n})$, $v_2(\boldsymbol{n})$, and $v_3(\boldsymbol{n})$, say; and let $\theta$, $\varphi$ be the directing angles of $\boldsymbol{n}$ so that $\boldsymbol{n} = [\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi]^T$. If $\theta$ and $\varphi$ span the intervals $[0, 2\pi[$ and $[0, \pi[$, respectively, then the equations

$$v = v_1(\boldsymbol{n}), \qquad v = v_2(\boldsymbol{n}), \qquad v = v_3(\boldsymbol{n}), \tag{2.31}$$

define three sheets in the ($v$, $\theta$, $\phi$)-space (spherical coordinates). Any vector with its origin at $O$ and its tip on one of these sheets is a velocity vector $\boldsymbol{v} = v\boldsymbol{n}$ for one of the three waves propagating in the $\boldsymbol{n}$ direction. Equations (2.31) define the *phase velocity surface*.

Now recall from (2.13) that $\boldsymbol{Q}(\boldsymbol{n})$ is quadratic in $\boldsymbol{n}$. It follows then from the secular equation (2.16) that the functions $v_1(\boldsymbol{n})$, $v_2(\boldsymbol{n})$, and $v_3(\boldsymbol{n})$ are homogeneous of degree one in $\boldsymbol{n}$. Dividing (2.31) across by $v$ gives the equations

$$1 = v_1(\boldsymbol{s}), \qquad 1 = v_2(\boldsymbol{s}), \qquad 1 = v_3(\boldsymbol{s}), \tag{2.32}$$

which define three sheets in the ($s$, $\theta$, $\phi$)-space (spherical coordinates). Any vector with its origin at $O$ and its tip on one of these sheets is a slowness vector $\boldsymbol{s} = s\boldsymbol{n} = v^{-1}\boldsymbol{n}$ for one of the three waves propagating in the $\boldsymbol{n}$ direction. Equations (2.32) define the *slowness surface*.

When two sheets of the phase velocity surface (or equivalently, of the slowness surface) intersect, the secular equation has a double root and thus the corresponding $\boldsymbol{n}$ gives the direction of an acoustical axis.

Now multiply (2.32) across by the frequency to get $\omega = v_\alpha(\boldsymbol{k})$ ($\alpha = 1, 2, 3$). Differentiating with respect to the wave vector $\boldsymbol{k} = k\boldsymbol{n}$ gives the group velocity of (1.9) as

$$\boldsymbol{v}_g = \frac{\partial v_\alpha(\boldsymbol{k})}{\partial \boldsymbol{k}} = \frac{\partial v_\alpha(\boldsymbol{s})}{\partial \boldsymbol{s}}. \tag{2.33}$$

The equations

$$\boldsymbol{v}_g = \frac{\partial v_1(\boldsymbol{s})}{\partial \boldsymbol{s}}, \qquad \boldsymbol{v}_g = \frac{\partial v_2(\boldsymbol{s})}{\partial \boldsymbol{s}}, \qquad \boldsymbol{v}_g = \frac{\partial v_3(\boldsymbol{s})}{\partial \boldsymbol{s}}, \tag{2.34}$$

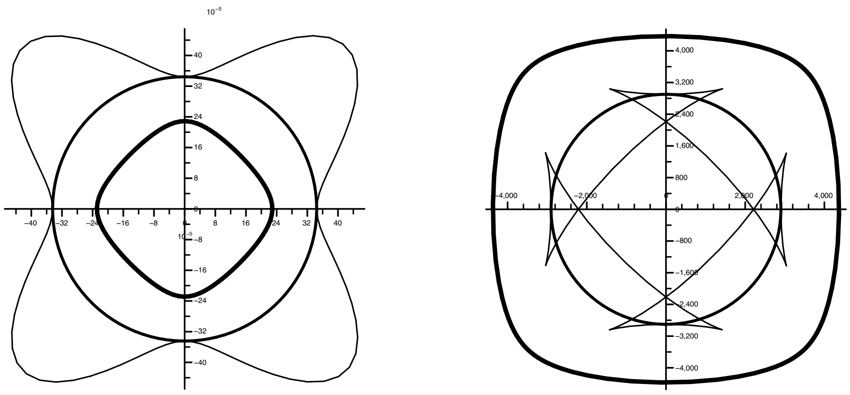


Figure 3: Homogeneous bulk waves in a plane of symmetry for copper (cubic symmetry). On the left: slowness surface (s/m); On the right: wavefronts, represented by the group velocity (m/s). Values used: $d_{11} = 17.0$, $d_{12} = 12.0$, $d_{44} = 7.5$ ($10^{10}$ Pa), and $\rho = 8.9 \times 10^3$ (kg/m$^3$).

define the group velocity. Because the energy travels at velocity $\boldsymbol{v}_g$, the contours traced by the tip of these vectors as $\boldsymbol{n}$ spans the unit sphere give a representation of the *wavefronts*, which can actually be observed using optical interferometry. These equations also show that for a given $\boldsymbol{n}$, the group velocity is normal to the slowness surface, which explains why experimentalists prefer slowness surfaces to phase velocity surfaces. The preferred directions of wavefront expansion correspond to a concentration of energy flux, and are called *caustics*.

Figure 3 displays the intersection of the slowness surface and of the wavefronts with the symmetry plane $\varphi = 0$ in Copper. There the slowness surface sheets intersect each other only at $\theta = 0, \pi/2$, showing that the symmetry axes are acoustic axes.

In *isotropic* solids, the phase velocity surface and the slowness surface consist of spherical sheets, and the two sheets related to the transverse waves coincide. The wavefronts expand spherically.

## 2.4 Inhomogeneous waves in isotropic solids

Consider the case where the displacement vector field is given by (1.8). The eigenvalue problem emerging from the equations of motion is

$$\boldsymbol{Q}(\boldsymbol{S})\boldsymbol{A} = \rho \boldsymbol{A}, \qquad \text{with} \qquad \boldsymbol{Q}(\boldsymbol{S}) = (\lambda + \mu)\boldsymbol{S} \otimes \boldsymbol{S} + \mu(\boldsymbol{S} \cdot \boldsymbol{S})\boldsymbol{I}. \quad (2.35)$$

A systematic procedure for finding the solutions can be established by writing the slowness as

$$\boldsymbol{S} = N\boldsymbol{C}, \qquad \text{where} \qquad \boldsymbol{C} = m\hat{\boldsymbol{m}} + \mathrm{i}\hat{\boldsymbol{n}}, \qquad N = T\mathrm{e}^{\mathrm{i}\phi}, \tag{2.36}$$

Here $\boldsymbol{C}$ is called the complex propagation vector, $m \geqslant 1$ is a real number, and $N$ is the complex scalar slowness, with amplitude $T$ and phase $\phi$; also, $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$ are orthogonal unit vectors, arbitrarily prescribed. Computing the real part $\boldsymbol{S}^+$ and the imaginary part $\boldsymbol{S}^-$ of the slowness, we find that

$$\frac{\boldsymbol{S}^+ \cdot \boldsymbol{S}^-}{|\boldsymbol{S}^+ \times \boldsymbol{S}^-|} = \frac{m^2 - 1}{2m} \sin 2\phi, \tag{2.37}$$

showing that the planes of constant phase are orthogonal to the planes of constant amplitude when $m = 1$ or when $\phi = 0, \pi$.

With this convention, the propagation condition (2.35) becomes

$$\boldsymbol{Q}(\boldsymbol{C})\boldsymbol{A} = \rho N^{-2}\boldsymbol{A}, \qquad \text{with} \qquad \boldsymbol{Q}(\boldsymbol{C}) = (\lambda + \mu)\boldsymbol{C} \otimes \boldsymbol{C} + \mu(m^2 - 1)\boldsymbol{I}. \tag{2.38}$$

The secular equation: det $[\boldsymbol{Q}(\boldsymbol{C}) - \rho N^{-2}\boldsymbol{I}] = 0$ factorizes and gives a simple eigenvalue

$$\rho N_L^{-2} = (\lambda + 2\mu)(m^2 - 1), \tag{2.39}$$

and a double eigenvalue

$$\rho N_T^{-2} = \mu(m^2 - 1). \tag{2.40}$$

Both are real, so that the phases $\phi_L$ and $\phi_T$ of $N_L$ and $N_T$ are zero: the corresponding inhomogeneous waves have orthogonal planes of constant phase and planes of constant amplitude.

Take $\boldsymbol{A} = \boldsymbol{C}$. It is clearly an eigenvector of $\boldsymbol{Q}(\boldsymbol{C})$ with eigenvalue $\rho N_L^{-2}$. The corresponding wave is

$$\boldsymbol{u} = \mathrm{e}^{-\omega N_L \hat{\boldsymbol{n}} \cdot \boldsymbol{x}} \left\{ (m\hat{\boldsymbol{m}} + \mathrm{i}\hat{\boldsymbol{n}}) \exp \mathrm{i}\omega(mN_L\hat{\boldsymbol{m}} \cdot \boldsymbol{x} - t) \right\}^+ . \tag{2.41}$$

On the other hand, the medium is unbounded and isotropic so that, without loss of generality, $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$ may be prescribed along two coordinate axes $x_1$ and $x_2$, say. Hence the following mechanical displacement

$$\begin{aligned} u_1 &= m\mathrm{e}^{-\omega N_L x_2} \cos \omega(mN_L x_1 - t), \\ u_2 &= -\mathrm{e}^{-\omega N_L x_2} \sin \omega(mN_L x_1 - t), \\ u_3 &= 0, \end{aligned} \tag{2.42}$$

is a solution to the equations of elastodynamics in an infinite isotropic solid. It represents an inhomogeneous wave, propagating with speed $(mN_L)^{-1}$ in the $x_1$ direction and attenuated with factor $\omega N_L$ in the $x_2$ direction. Here $N_L$ is given by (2.39), and $\omega$ and $m > 1$ are arbitrary ($m$ cannot be equal to 1 according to (2.39); this also precludes the wave from being circularly polarized, according to (2.42).)

Next take $\boldsymbol{A}$ to be any complex vector such that $\boldsymbol{A} \cdot \boldsymbol{C} = 0$. It is clearly an eigenvector of $\boldsymbol{Q}(\boldsymbol{C})$ corresponding to the eigenvalue $\rho N_T^{-2}$. Again without loss of generality, we take $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$ aligned with the coordinate axes $x_1$ and $x_2$, so that $\boldsymbol{C} = [m, \mathrm{i}, 0]^T$. Then $\boldsymbol{A} = [1, \mathrm{i}m, \gamma]^T$ where $\gamma$ is arbitrary, and its multiples, are the most general vectors such that $\boldsymbol{A} \cdot \boldsymbol{C} = 0$. The corresponding solution is

$$\begin{aligned} u_1 &= \mathrm{e}^{-\omega N_T x_2} \cos \omega(mN_T x_1 - t), \\ u_2 &= -m\mathrm{e}^{-\omega N_T x_2} \sin \omega(mN_T x_1 - t), \\ u_3 &= \mathrm{e}^{-\omega N_T x_2}[\gamma^+ \cos \omega(mN_T x_1 - t) - \gamma^- \sin \omega(mN_T x_1 - t)]. \end{aligned} \tag{2.43}$$

It represents an inhomogeneous wave, propagating with speed $(mN_T)^{-1}$ in the $x_1$ direction and attenuated with factor $\omega N_T$ in the $x_2$ direction. Here $N_T$ is given by (2.40), and $\omega$, $\gamma^+$, $\gamma^-$, and $m > 1$ are arbitrary ($m$ cannot be equal to 1 according to (2.39)). Note that circular polarization corresponds to $\gamma^+ = \sqrt{m^2 - 1}$, $\gamma^- = 0$.

# 3 Surface waves

We solve what is arguably the simplest boundary value problem of elastodynamics: the propagation of a wave in a half-space. Hence we must solve the equations of motion for a wave traveling over the flat surface of a semi-infinite solid. This set-up of course originates from the need to model *seismic waves*.

We begin with the simplest model possible and take an isotropic, linearly elastic, homogeneous solid to occupy the half-space $x_2 \geqslant 0$, say. The surface of the solid is assumed to be free of traction. Hence

$$\sigma_{j2} = 0 \qquad \text{at } x_2 = 0 \tag{3.1}$$

are the *boundary conditions*.

## 3.1 Shear horizontal homogeneous surface waves

Our first candidates for a wave solution are *homogeneous plane waves*,

$$\boldsymbol{u}(x_1, x_2, x_3, t) = \boldsymbol{U}^{\boldsymbol{0}} \mathrm{e}^{\mathrm{i}k(\boldsymbol{x} \cdot \boldsymbol{n} - vt)}, \tag{3.2}$$

where $\boldsymbol{U^0} = [U_1^0, U_2^0, U_3^0]^t$ is the amplitude vector, $\boldsymbol{n} = [n_1, 0, n_3]^t$ is the unit vector in the direction of propagation, $k$ is the wave number, and $v$ is the speed. Because the solid is isotropic, every direction of propagation is equivalent; we choose $\boldsymbol{n} = [1, 0, 0]^t$ for instance. From the constitutive relations it follows that $\boldsymbol{\sigma}$ is of the same form as $\boldsymbol{u}$,

$$\boldsymbol{\sigma}(x_1, x_2, x_3, t) = \mathrm{i}k\boldsymbol{T^0}\mathrm{e}^{\mathrm{i}k(x_1 - vt)}, \tag{3.3}$$

say, where the $T_{ij}^0$ are constants. Now the stress-strain relations (2.7) give

$$\begin{aligned} T_{11}^0 &= (\lambda + 2\mu)U_1^0, & T_{12}^0 &= \mu U_2^0, & T_{13}^0 &= \mu U_3^0, \\ T_{22}^0 &= \lambda U_1^0, & T_{32}^0 &= 0, & T_{33}^0 &= 0. \end{aligned} \tag{3.4}$$

The equations of motion (2.4) reduce to

$$T_{11}^0 = \rho v^2 U_1^0, \qquad T_{12}^0 = 0, \qquad T_{13}^0 = 0. \tag{3.5}$$

Finally the boundary conditions (3.1) are here

$$T_{12}^0 = 0, \qquad T_{22}^0 = 0, \qquad T_{32}^0 = 0. \tag{3.6}$$

These sets of equations are compatible only when

$$U_1^0 = 0, \qquad U_2^0 = 0, \qquad U_3^0 \neq 0, \qquad \rho v^2 = \mu. \tag{3.7}$$

This is called a homogeneous *shear horizontal surface wave* (SH surface wave). It is linearly polarized, in the direction parallel to the free surface and transverse with respect to the direction of propagation. It travels at speed $v = \sqrt{\mu/\rho}$, with an arbitrary wave number $k$ (hence it is not dispersive). It oscillates with amplitude $U_3^0$ in the plane parallel to the surface, at frequency $\omega = kv$, see Figure 4. Note that it penetrates into the substrate with an amplitude $U_3^0$ which remains constant ad infinitum. However, seismic waves in general travel for long distances – sometimes several times around the globe – and their amplitude variations are clearly localized near the surface. We thus need to refine our type of solution in order to model this confined behavior.

Moreover, as we shall soon discover, non-decaying homogeneous surface waves are unlikely to exist at all in reality, because a slight change in the boundary conditions or properties of the media turns it into a decaying (localized) solution.

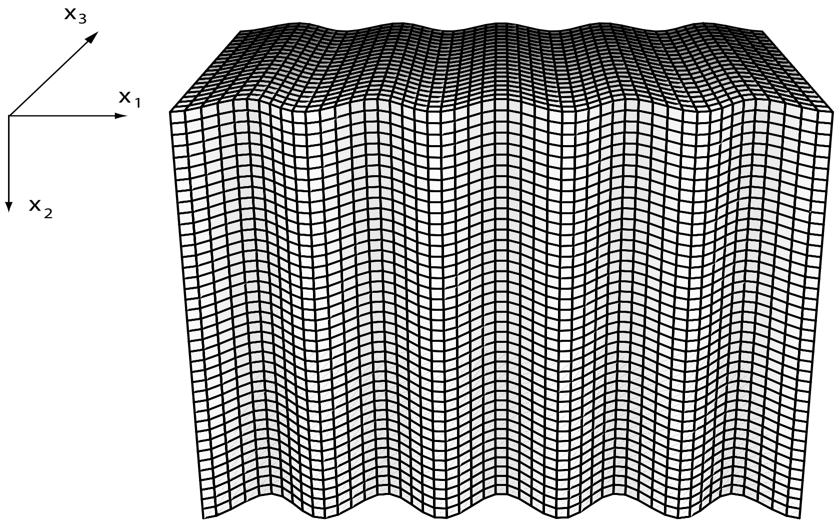


Figure 4: A shear horizontal surface wave in a homogeneous semi-infinite solid. It propagates in the $x_1$-direction, and is polarized in the $x_3$-direction. Its amplitude remains constant with the depth $x_2$. It is unlikely to be observed at all.

### 3.2 Rayleigh waves

We now look for a solution to the boundary value problem (2.4), (3.1) in the form of a wave with an amplitude depending on the depth,

$$\boldsymbol{u}(x_1, x_2, x_3, t) = \boldsymbol{U}(\mathrm{i}kx_2)\mathrm{e}^{\mathrm{i}k(x_1 - vt)}, \qquad \boldsymbol{\sigma}(x_1, x_2, x_3, t) = \mathrm{i}k\boldsymbol{T}(\mathrm{i}kx_2)\mathrm{e}^{\mathrm{i}k(x_1 - vt)}, \tag{3.8}$$

where $\boldsymbol{U}$ and $\boldsymbol{T}$ are functions of the variable $\mathrm{i}kx_2$ only. These are *inhomogeneous* plane waves. They are called two-dimensional waves, because their variations depend on two space variables only ($x_1$ and $x_2$ here). This character allows us, in contrast with SH surface waves, to look for an amplitude which decays with depth:

$$\boldsymbol{U} \to \boldsymbol{0}, \qquad \boldsymbol{T} \to \boldsymbol{0}, \qquad \text{as} \qquad x_2 \to \infty. \tag{3.9}$$

Now the stress-strain relations (2.7) give

$$\begin{array}{lll} T_{11} = (\lambda + 2\mu)U_1 + \lambda U_2', & T_{12} = \mu U_2 + \mu U_1', & T_{13} = \mu U_3, \\ T_{22} = \lambda U_1 + (\lambda + 2\mu)U_2', & T_{32} = \mu U_3', & T_{33} = 0, \end{array} \tag{3.10}$$

where the prime denotes differentiation with respect to the variable $\mathrm{i}kx_2$. The equations of motion (2.4) then reduce to

$$T_{11} + T_{12}' = \rho v^2 U_1, \qquad T_{12} + T_{22}' = \rho v^2 U_2, \qquad T_{13} + T_{32}' = \rho v^2 U_3. \tag{3.11}$$

Clearly the equations involving $T_{11}$, $T_{12}$, $T_{22}$, and $U_1$, $U_2$ (in-plane motion) are decoupled from those involving $T_{13}$, $T_{32}$, and $U_3$ (anti-plane motion).

First we consider the anti-plane equations: they combine to give $\mu U_3'' + (\mu - \rho v^2)U_3 = 0$, or equivalently, $\mu T_{32}'' + (\mu - \rho v^2)T_{32} = 0$. The general solution to this latter equation is

$$T_{32}(\mathrm{i}kx_2) = A\mathrm{e}^{\sqrt{1-\rho v^2/\mu}\,kx_2} + B\mathrm{e}^{-\sqrt{1-\rho v^2/\mu}\,kx_2}, \tag{3.12}$$

where $A$ and $B$ are constants. Because of (3.9), the square root in the arguments must be real, and then we must take $A = 0$ to ensure decay. On the other hand, $T_{32}(0) = 0$ so that $B = 0$ also. Hence there are no anti-plane inhomogeneous (decaying) surface waves in an isotropic elastic homogeneous solid.

Now we consider the remaining in-plane equations. They can be rearranged as follows

$$\begin{bmatrix} U_1' \\ U_2' \\ T_{12}' \\ T_{22}' \end{bmatrix} = \begin{bmatrix} 0 & -1 & \dfrac{1}{\mu} & 0 \\ -\dfrac{\lambda}{\lambda+2\mu} & 0 & 0 & \dfrac{1}{\lambda+2\mu} \\ \rho v^2 - 4\dfrac{\mu(\lambda+\mu)}{\lambda+2\mu} & 0 & 0 & -\dfrac{\lambda}{\lambda+2\mu} \\ 0 & \rho v^2 & -1 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ T_{12} \\ T_{22} \end{bmatrix}, \tag{3.13}$$

or, in matrix notation,

$$\boldsymbol{\eta}' = N\boldsymbol{\eta}, \qquad \text{where} \qquad \boldsymbol{\eta} \equiv [U_1, U_2, T_{12}, T_{22}]^t, \tag{3.14}$$

is the in-plane displacement-traction vector, and $N$ is the $4 \times 4$ matrix above. Clearly this equation has an exponential solution,

$$\boldsymbol{\eta}(\mathrm{i}kx_2) = \mathrm{e}^{\mathrm{i}kqx_2}\boldsymbol{\eta}^{(0)}, \tag{3.15}$$

say, where $\boldsymbol{\eta}^{(0)}$ is a constant vector and $q$ is the decay. To enforce (3.9), $q$ must be such that

$$\mathrm{Im}(q) > 0. \tag{3.16}$$

Using (3.15) leads to the resolution of an eigenvalue problem: $(N - qI)\boldsymbol{\eta}^{(0)} = \mathbf{0}$. The corresponding characteristic equation: $\det\,(N - qI) = 0$, is called the *propagation condition*. Here it factorizes into

$$\left(q^2 + 1 - \frac{v^2}{v_T^2}\right)\left(q^2 + 1 - \frac{v^2}{v_L^2}\right) = 0, \tag{3.17}$$

where $v_T \equiv \sqrt{\mu/\rho}$ and $v_L \equiv \sqrt{(\lambda + 2\mu)/\rho}$ are the speeds of the shear and longitudinal homogeneous bulk waves. For $q$ to have a non-zero imaginary part, $v$ must be less than $v_T$ (and thus less than $v_L$),

$$0 < v < v_T < v_L. \tag{3.18}$$

These inequalities define the *subsonic range*, and are consistent with observations. It has indeed been recorded experimentally that seismic surface waves are slower than bulk waves. At the same time, their energy remains confined near the free surface due to the rapid depth decay, so that they travel further, and create more damage than bulk (homogeneous) waves, away from the epicenter.

The eigenvalues are $q = q_1, q_2$, defined by

$$q_1 = \mathrm{i}\sqrt{1 - \frac{v^2}{v_T^2}}, \qquad q_2 = \mathrm{i}\sqrt{1 - \frac{v^2}{v_L^2}}, \tag{3.19}$$

and the corresponding eigenvectors are $\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}$, defined by

$$\boldsymbol{\eta}^{(1)} = \begin{bmatrix} \sqrt{1 - \frac{v^2}{v_T^2}} \\ \mathrm{i} \\ \mathrm{i}\mu\left(2 - \frac{v^2}{v_T^2}\right) \\ -2\mu\sqrt{1 - \frac{v^2}{v_T^2}} \end{bmatrix}, \qquad \boldsymbol{\eta}^{(2)} = \begin{bmatrix} 1 \\ \mathrm{i}\sqrt{1 - \frac{v^2}{v_L^2}} \\ 2\mathrm{i}\mu\sqrt{1 - \frac{v^2}{v_L^2}} \\ -\mu\left(2 - \frac{v^2}{v_T^2}\right) \end{bmatrix}. \tag{3.20}$$

Then the general solution of the inhomogeneous form (3.8), satisfying the decay condition (3.16), propagating in the semi-infinite solid, is

$$\boldsymbol{\eta} = c_1 \mathrm{e}^{-k\sqrt{1 - \frac{v^2}{v_T^2}}\,x_2} \boldsymbol{\eta}^{(1)} + c_2 \mathrm{e}^{-k\sqrt{1 - \frac{v^2}{v_L^2}}\,x_2} \boldsymbol{\eta}^{(2)}, \tag{3.21}$$

where $c_1$ and $c_2$ are yet arbitrary constants. For a wave leaving the boundary $x_2 = 0$ free of traction, we have according to (3.1), $\boldsymbol{\eta}(0) = [U_1(0), U_2(0), 0, 0]^t$ so that

$$\mu \begin{bmatrix} \mathrm{i}\left(2 - \frac{v^2}{v_T^2}\right) & 2\mathrm{i}\sqrt{1 - \frac{v^2}{v_L^2}} \\ -2\sqrt{1 - \frac{v^2}{v_T^2}} & -\left(2 - \frac{v^2}{v_T^2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.22}$$

The associated determinantal equation is

$$\left(2-\frac{v^2}{v_T^2}\right)^2-4\sqrt{1-\frac{v^2}{v_T^2}}\sqrt{1-\frac{v^2}{v_L^2}}=0, \tag{3.23}$$

and is called the *secular equation for Rayleigh surface waves* (named after Lord Rayleigh, who solved this problem in 1885.) It gives implicitly the speed in terms of $v_L=\sqrt{(\lambda+2\mu)/\rho}$ and $v_T=\sqrt{\mu/\rho}$. This speed is independent of the wave number $k$ and Rayleigh waves are thus non-dispersive. The *exact Rayleigh equation* (3.23) can be squared to give

$$f(\eta)\equiv\eta^2\left[\eta^6-8\eta^2+\left(24-16\frac{v_T^2}{v_L^2}\right)\eta^2+16\left(\frac{v_T^2}{v_L^2}-1\right)\right]=0, \qquad \eta\equiv\frac{v}{v_T}. \tag{3.24}$$

This equation is the product of $\eta^2$ (which gives $v=0$, a *spurious* root) and of a cubic in $\eta^2$, the so-called *Rayleigh cubic equation*, which can be proved to have a single root in the subsonic range, coinciding with the root of (3.23). In terms of $\eta=v/v_T$ and $\nu$, the Poisson ratio, the exact and cubic Rayleigh equations read

$$(2-\eta^2)^2-4\sqrt{1-\eta^2}\sqrt{1-\frac{1-2\nu}{2-2\nu}\eta^2}=0, \tag{3.25}$$

and

$$(1-\nu)\eta^6-8(1-\nu)\eta^4+8(2-\nu)\eta^2-8=0, \tag{3.26}$$

respectively.

It is a simple matter to solve numerically either equation for the speed, see the diagonal of Table 2 for Silica, Aluminum, Iron, Steel, and Nickel. A popular approximate expression is

$$\eta\equiv\frac{v}{v_T}\simeq\frac{0.87+1.12\nu}{1+\nu}, \tag{3.27}$$

with a relative error of less than 0.5 in the $0\leqslant\nu\leqslant 0.5$ range, see Figure 5. For instance, rocks are often considered to have Poisson ratio $\nu=1/4$ (and then $\lambda=\mu$ and $v_T^2/v_L^2=1/3$). In that case, Rayleigh's equations have an exact solution, $\eta=\sqrt{2-2/\sqrt{3}}\simeq 0.91940$ and the approximation above gives $\eta=0.92$. The proof for (3.27) relies on the observation that $\eta$ is close to 1 in the range $0\leqslant\nu\leqslant 0.5$. Writing $\eta\simeq 1-\delta$ where $\delta$ is small, we find from (3.24) that $\delta=f(1)/f'(1)$ so that $\eta\simeq 1-f(1)/f'(1)$. Explicitly, $f(1)=-(1-\nu)$, $f'(1)=-8(1+\nu)$ and $\eta\simeq(7/8+9\nu/8)/(1+\nu)$, which is (3.27), once the last digit in the fractions has been truncated.

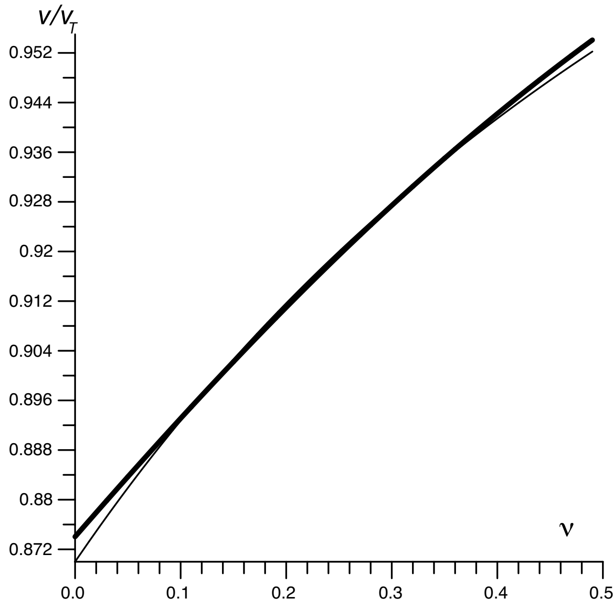


Figure 5: The speed of a Rayleigh wave in an isotropic solid with Poisson ratio $\nu$. Thick curve: numerical solution to the exact equation; Thin curve: approximate solution using the formula $v/v_T \simeq (0.87 + 1.12\nu)/(1+\nu)$.

Once the speed is computed, the complete description of the Rayleigh wave follows: hence, either line of (3.22) gives the ratio $c_1/c_2$ (which turns out to be real here) and (3.21) gives the displacements and tractions. Hence we find that the displacements are proportional to

$$u_1 = \left[ \mathrm{e}^{-\alpha_L k x_2} - \frac{2\alpha_L \alpha_T}{1+\alpha_T^2} \mathrm{e}^{-k\alpha_T x_2} \right] \cos k(x_1 - vt),$$
$$u_2 = -\alpha_L \left[ \mathrm{e}^{-\alpha_L k x_2} - \frac{2}{1+\alpha_T^2} \mathrm{e}^{-k\alpha_T x_2} \right] \sin k(x_1 - vt), \qquad (3.28)$$

where

$$\alpha_L \equiv \sqrt{1 - \frac{v^2}{v_L^2}}, \qquad \alpha_T \equiv \sqrt{1 - \frac{v^2}{v_T^2}}, \qquad (3.29)$$

are coefficients of attenuation. These expressions clearly show the "vertical" elliptical polarization and the exponential decay with depth. In many respects Rayleigh surface waves resemble water surface waves, with the main differences that they are sustained by elasticity alone —not gravity— and that the ellipses of polarization are described near the surface in a retrograde —not direct— manner with respect to the propagation. This latter

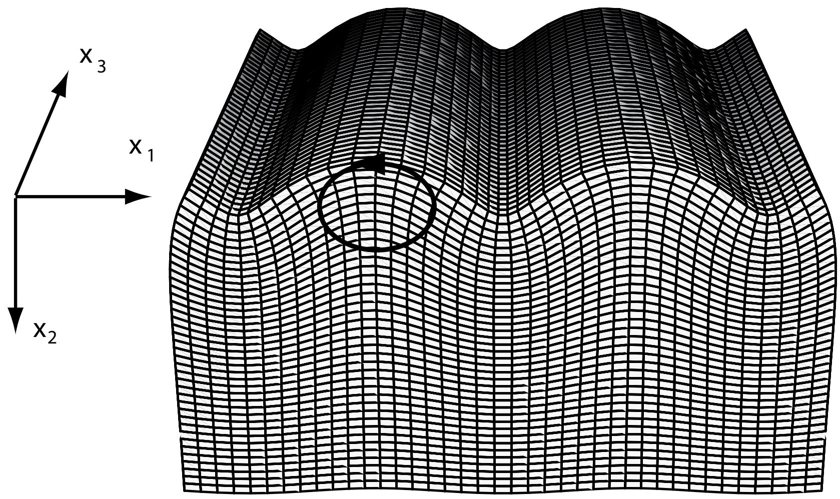


Figure 6: A Rayleigh surface wave in a homogeneous semi-infinite solid with Poisson ratio $\nu = 1/4$. Its amplitude decays exponentially with depth and is polarized in the sagittal plane (which includes the direction of propagation $x_1$ and the normal to the boundary $x_2$). The ellipse of polarization is described in a retrograde manner near the surface.

property is quickly verified by considering the surface displacements at the time $t = x_1/v$ (where we find $u_1 > 0$, $u_2 = 0$) and then at the later time $t = x_1/v + \pi/(2kv)$ (where we find $u_1 = 0$, $u_2 < 0$). Figure 6 shows the displacements incurred by a Rayleigh surface wave in an isotropic homogeneous solid with $\lambda = \mu$.

## 3.3 Love waves

Rayleigh waves account for seismic waves which do not penetrate too deeply into the crust and have a vertical elliptical polarization. However, other types of localized waves are also recorded during a seismic event; in particular, shear horizontal dispersive waves, faster than the Rayleigh wave. Since a homogeneous substrate cannot accommodate such a surface wave, see above, A.E.H. Love thought of considering the effect of what can be considered the simplest inhomogeneous set-up for surface waves, namely the superposition of a layer made of an isotropic elastic solid on top of a half-space made of another isotropic elastic solid.

Hence the layer is in the region $-h \leqslant x_2 \leqslant 0$, and is filled with a solid with mass density $\hat{\rho}$ and shear modulus $\hat{\mu}$ say, defining the shear bulk wave speed $\hat{v}_T \equiv \sqrt{\hat{\mu}/\hat{\rho}}$. The transverse displacement $\hat{U}_3$ and traction $\hat{T}_{32}$ satisfy equations similar to (3.10)-(3.11); in particular,

$$\hat{U}_3'' + (1 - v^2/\hat{v}_T^2)\hat{U}_3 = 0. \tag{3.30}$$

First we consider the case where the wave, if it exists, travels at a speed which is *supersonic* with respect to the layer:

$$v > \hat{v}_T \equiv \sqrt{\hat{\mu}/\hat{\rho}}. \tag{3.31}$$

Then the solution to (3.30) has sinusoidal variations,

$$\hat{U}_3(\mathrm{i}kx_2) = \hat{U}_3(0)\left[\cos\sqrt{v^2/\hat{v}_T^2 - 1}\;kx_2 - C\sin\sqrt{v^2/\hat{v}_T^2 - 1}\;kx_2\right], \tag{3.32}$$

where $C$ is a constant. It follows that $\hat{T}_{32} = \hat{\mu}\hat{U}_3'$ is given by

$$\hat{T}_{32} = \mathrm{i}\hat{\mu}\sqrt{v^2/\hat{v}_T^2 - 1}\;\hat{U}_3(0)\left[\sin\sqrt{v^2/\hat{v}_T^2 - 1}\;kx_2 + C\cos\sqrt{v^2/\hat{v}_T^2 - 1}\;kx_2\right]. \tag{3.33}$$

The surface wave leaves the upper face of the superficial layer free of traction: $\hat{T}_{32} = 0$ at $x_2 = -h$, which determines the constant as $C = \tan\sqrt{v^2/\hat{v}_T^2 - 1}\;kh$.

Next, in the semi-infinite substrate, we must have an exponentially decaying solution to the equation of motion $U_3'' + (v^2/v_T^2 - 1)U_3 = 0$, in order to satisfy (3.9). This is possible when the wave is subsonic with respect to the substrate, as in (3.18). Hence we find in turn that

$$U_3 = U_3(0)\mathrm{e}^{-\sqrt{1-v^2/v_T^2}\;kx_2}, \qquad T_{32} = \mathrm{i}\mu\sqrt{1 - v^2/v_T^2}\;U_3(0)\mathrm{e}^{-\sqrt{1-v^2/v_T^2}\;kx_2}, \tag{3.34}$$

Now, at the layer/substrate interface, the bond is *rigid*, so that displacements and tractions are continuous:

$$\hat{U}_3(0) = U_3(0), \qquad \hat{T}_{32}(0) = T_{32}(0). \tag{3.35}$$

These conditions reduce *in fine* to a single equation,

$$\hat{\mu}\sqrt{v^2/\hat{v}_T^2 - 1}\;\tan\sqrt{v^2/\hat{v}_T^2 - 1}\;kh - \mu\sqrt{1 - v^2/v_T^2} = 0, \tag{3.36}$$

the *dispersion equation for Love waves.* This wave is *dispersive* because its speed obviously depends on the wave-number. Moreover, several modes of propagation may exist, due to the periodic nature of the tan function.

First, consider the limit $kh \to 0$: then the layer disappears and $v$ tends to $v_T = \sqrt{\mu/\rho}$, the bulk shear wave speed in the substrate. As $kh$ increases, the dispersion equation gives a speed $v$ which goes from $v_T$ (at $kh = 0$) to

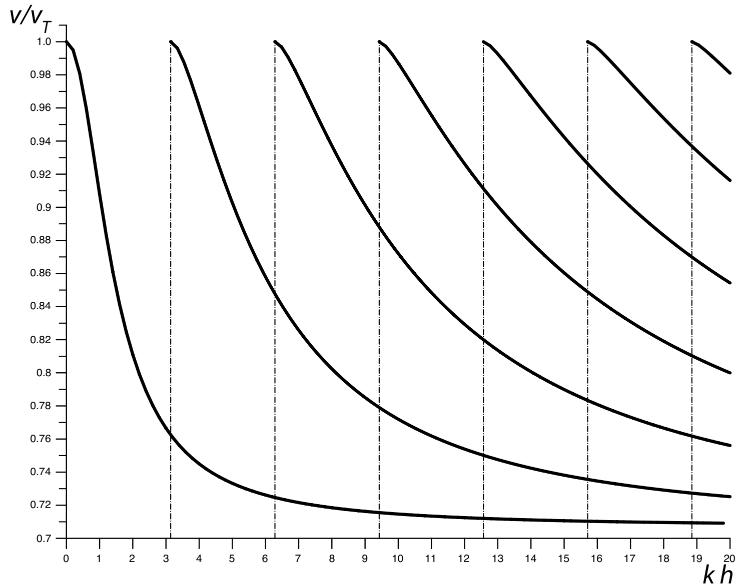


Figure 7: Dispersion curves for a Love surface wave in a superficial layer with shear modulus $\hat{\mu}$ and mass density $\hat{\rho}$, rigidly bonded to a semi-infinite substrate with shear modulus $\mu = 2\hat{\mu}$ and mass density $\rho = \hat{\rho}$. Several modes of propagation may exist for a given thickness to wavelength ratio. For low values of $kh$ where $k$ is the wave-number and $h$ the layer thickness, only the fundamental mode exists; here, new modes of propagation appear each time $kh$ is an integer multiple of $\pi$.

$\hat{v}_T = \sqrt{\hat{\mu}/\hat{\rho}}$ (at $kh = \infty$), following the graph of the so-called *fundamental mode of propagation*. Next consider the limit $kh \to \pi/\sqrt{v_T^2/\hat{v}_T^2 - 1}$: again $v = v_T$ is a root to the dispersion equation. The graph going from $v_T$ (as $kh = \pi/\sqrt{v_T^2/\hat{v}_T^2 - 1}$) to $\hat{v}_T$ (at $kh = \infty$) is that of the mode 1 of propagation. This analysis can be repeated as $kh \to n\pi/\sqrt{v_T^2/\hat{v}_T^2 - 1}$ where $n$ is an integer, defining the starting point of the graph for the *n-th mode of propagation*.

To draw Figure 7, we take a structure such that $\hat{\mu} = \mu/2$ and $\hat{\rho} = \rho$, so that $v_T = \sqrt{2}\ \hat{v}_T$, and so that the graph for the $n$-th mode of propagation starts at $kh = n\pi$. To display an example of Love wave displacements, we focus on a wave with wavelength equal to 4 times the layer thickness: $kh = \pi/2$. At that value, only the fundamental mode propagates, see Figure 7. We find numerically that it propagates with speed $v \simeq 0.8442\ v_T$. Then $C \simeq 1.644$ and the mechanical fields (3.32)-(3.34) are entirely determined, see Figure 8.

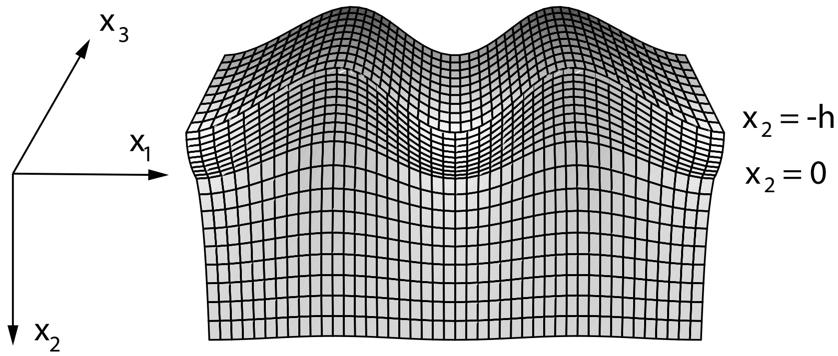


Figure 8: Displacements due to a Love surface wave in a superficial layer with shear modulus $\hat{\mu}$ and mass density $\hat{\rho}$, rigidly bonded to a semi-infinite substrate with shear modulus $\mu = 2\hat{\mu}$ and mass density $\rho = \hat{\rho}$. Here the wave propagates with a wavelength which is 4 times $h$, the layer thickness, and only the fundamental mode exists.

Compiling (3.9) and (3.31) we find the following range of existence for the Love wave speed,

$$\sqrt{\hat{\mu}/\hat{\rho}} = \hat{v}_T < v < v_T = \sqrt{\mu/\rho}. \tag{3.37}$$

It is often said that Love waves occur in the case of a "soft", or "slow" superficial layer over a "hard", or "fast" substrate.

In general, even a very thin layer produces a Love wave, which is a *decaying* shear horizontal surface wave. In that respect it is fair to say that the shear horizontal homogeneous (non-decaying) surface waves of Section 3.1 are unlikely to be observed as such, because a slight heterogeneity near the surface causes it to decay. It can be shown also that if the boundary is not exactly plane, then a localization of the amplitude is also observed.

## 3.4 Other surface waves

Love also studied the effect of a superficial layer on the propagation of Rayleigh waves. All sorts of combinations, effects, and refinements can be considered, including: several layers, viscoelasticity, gravity, coupled fields, roughness, curvature, pre-strains, residual stresses, etc.

## 3.5 Surface waves in anisotropic media

Anisotropy brings out new features in the analysis of surface wave propagation. Although the equations of motion seem almost impossible to solve in general, the boundary value problem of a Rayleigh wave (traction free boundary, amplitude decay with depth) has been completely resolved, thanks to a Hamiltonian formulation known as the *Stroh formalism.* It relies on advanced algebraic tools and the outline of the main results sketched below is brief and incomplete.

When the direction of propagation and the normal to the surface are both contained in a plane of symmetry, the Rayleigh wave is a two-component wave, like in isotropic solids.

If those two directions are aligned with the intersections of mutually orthogonal symmetry planes, then the propagation condition is a quadratic in $q^2$ and a complete resolution is possible, leading to an exact secular equation, which can be rationalized as a cubic in $v^2$. If those two directions are not aligned with the intersections of mutually orthogonal symmetry planes, then the propagation condition is a quartic in $q$. Again a complete resolution is possible, this time leading to a rationalized equation which is a quartic in $v^2$.

When the direction of propagation and the normal to the surface are not both contained in a plane of symmetry, the Rayleigh wave is a three-component wave.

If the free surface is a symmetry plane, then the propagation condition is a cubic in $q^2$ in general. For solids of cubic symmetry, the rationalized secular equation is a polynomial of degree 10 in $v^2$, and for solids of lesser symmetry, it is a polynomial of degree 12 in $v^2$. For propagation in other directions, the propagation condition is a sextic in $q$ in general, and the rationalized equation is a polynomial of degree 27 in $v^2$.

Of course these polynomials have many spurious roots (potentially 26!) and robust numerical procedures with a sure convergence towards the actual Rayleigh wave speed are often favored. Most of these methods rely on the concept of the *surface impedance matrix*, which relates tractions to displacements.

The polynomial secular equations are however useful when it comes to generating the slowness surface and its companion, the group velocity, because their differentiation with respect to the slowness (see (2.34)) is almost trivial. Figure 9 shows the results of such computations for a symmetry plane in (cubic) Copper. There is an almost perfect coincidence between the wavefronts generated mathematically and those generated by LASER impact (not shown here).

Note that a question has remained open for a long time: are there “forbidden” directions of propagation for Rayleigh waves in anisotropic solids? This question has now been answered; to cut a long story short, we may say that the existence of a unique Rayleigh wave is the rule, not the exception (and that their non-existence is the exception, not the rule.)

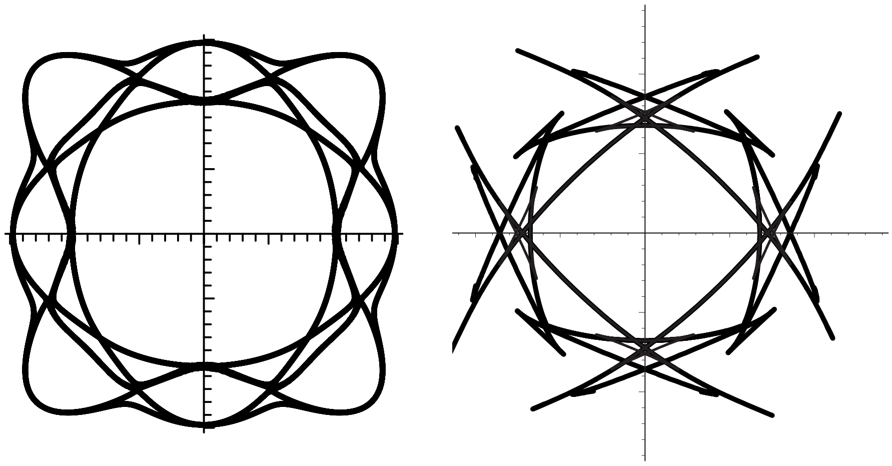

Figure 9: Surface waves in a plane of symmetry for Copper (cubic symmetry). On the left: slowness surface; On the right: wavefronts.

Similar comments apply to the other types of surface waves. One extremely active area of both fundamental and applied research is that linked to the coupling between elastic fields and electric fields through *piezoelectricity*. This property found in certain crystals actually *requires* anisotropy, as shown by the Curie brothers. This coupling allows for instance for the propagation of a pure SH wave which decays with depth, the so-called *Bleustein-Gulyaev wave*. Piezoelectric surface acoustic waves form the basic working principle of countless high-frequency signal devices such as global positioning systems, mobile phones, etc.

# 4 Interface waves

Surface waves occur at the interface between a solid and the vacuum. Elastic waves can also occur at the interface between two continua. We now consider in turn three important examples of such waves: Stoneley waves and slip waves —at the interface between two semi-infinite solids— and Scholte waves —at the interface between a semi-infinite solid and a perfect fluid.

## 4.1 Stoneley waves

Here the region $x_2 \geqslant 0$ is occupied by an isotropic solid of mass density $\rho$ and Lamé coefficients $\lambda$, $\mu$; and the region $x_2 \leqslant 0$ is occupied by another isotropic solid, of mass density $\hat{\rho}$ and Lamé coefficients $\hat{\lambda}$, $\hat{\mu}$. We look for a wave which respects a rigid bond between the two solids, with an amplitude which decays away from the interface $x_2 = 0$. In the $x_2 \geqslant 0$ region, the wave is described by (3.20)-(3.21). In the $x_2 \leqslant 0$ region, it is described by

$$\hat{\boldsymbol{\eta}} = \hat{c}_1 \mathrm{e}^{k\sqrt{1-\frac{v^2}{\hat{v}_T^2}}\,x_2}\hat{\boldsymbol{\eta}}^{(1)} + \hat{c}_2 \mathrm{e}^{k\sqrt{1-\frac{v^2}{\hat{v}_L^2}}\,x_2}\hat{\boldsymbol{\eta}}^{(2)}, \tag{4.1}$$

where $\hat{c}_1$ and $\hat{c}_2$ are constants, $\hat{v}_T = \sqrt{\hat{\mu}/\hat{\rho}}$, $\hat{v}_L = \sqrt{(\hat{\lambda}+\hat{\mu})/\hat{\rho}}$, and

$$\hat{\boldsymbol{\eta}}^{(1)} = \begin{bmatrix} \sqrt{1-\dfrac{v^2}{\hat{v}_T^2}} \\ -\mathrm{i} \\ -\mathrm{i}\hat{\mu}\left(2-\dfrac{v^2}{\hat{v}_T^2}\right) \\ -2\hat{\mu}\sqrt{1-\dfrac{v^2}{\hat{v}_T^2}} \end{bmatrix}, \qquad \hat{\boldsymbol{\eta}}^{(2)} = \begin{bmatrix} 1 \\ -\mathrm{i}\sqrt{1-\dfrac{v^2}{\hat{v}_L^2}} \\ -2\mathrm{i}\hat{\mu}\sqrt{1-\dfrac{v^2}{\hat{v}_L^2}} \\ -\hat{\mu}\left(2-\dfrac{v^2}{\hat{v}_T^2}\right) \end{bmatrix}. \tag{4.2}$$

At the interface $x_2 = 0$, the displacements and the tractions are continuous: $\boldsymbol{\eta}(0) = \hat{\boldsymbol{\eta}}(0)$, which gives a $4 \times 4$ determinantal equation for the constants $c_1$, $c_2$, $\hat{c}_1$, $\hat{c}_2$. It reads

$$\begin{vmatrix} \alpha_T & 1 & -\hat{\alpha}_T & -1 \\ 1 & \alpha_L & 1 & \hat{\alpha}_L \\ \mu\left(1+\alpha_T^2\right) & 2\mu\alpha_L & \hat{\mu}\left(1+\hat{\alpha}_T^2\right) & 2\hat{\mu}\hat{\alpha}_L \\ -2\mu\alpha_T & -\mu\left(1+\alpha_T^2\right) & 2\hat{\mu}\hat{\alpha}_T & \hat{\mu}\left(1+\hat{\alpha}_T^2\right) \end{vmatrix} = 0, \tag{4.3}$$

which is the *Stoneley wave equation.* Here the coefficients of attenuation $\alpha_T$ and $\alpha_L$ are defined in (3.29) and

$$\hat{\alpha}_L \equiv \sqrt{1-\frac{v^2}{\hat{v}_L^2}}, \qquad \hat{\alpha}_T \equiv \sqrt{1-\frac{v^2}{\hat{v}_T^2}}. \tag{4.4}$$

Like a Rayleigh wave, a Stoneley wave is non-dispersive; unlike a Rayleigh wave, there are serious restrictions to its existence, which depends strongly on the values of the material parameters. Surveys indicate that among the possible combinations between two isotropic real solids, less than 3% of the pairings let a Stoneley wave exist. Of course, this is not to say that for a given pair out of the remaining 97%, no elastic wave can propagate at all at the flat interface, but rather that if it does, then it must have characteristics other than those of the Stoneley wave type. For instance the bond might be slippery instead of rigid, giving rise to a so-called *slip wave*, see below.

The numbers above the diagonal in Table 2 give the speed of the Stoneley wave at the interface of pairs of isotropic solids taken among Silica, Aluminum, Iron, Steel, and Nickel, when it exists. Although the last digit is clearly irrelevant in view of the material data of Table 1, it is nonetheless given to highlight an important property of the Stoneley wave, which is that in order to decay away from the interface, it must travel at a speed which is subsonic with respect to the two solids. Hence the Stoneley wave speed at a Silica/Nickel interface is 2921 m/s, just below the shear bulk wave speed in Nickel, $v_T = 2922$ m/s, see Figure 10. Finally note that it has been proved that when it exists, a Stoneley wave travels with a speed which is larger than that of the slowest Rayleigh wave associated with the solids on either side of the interface.

## 4.2 Slip waves

Now imagine that the two solids are not perfectly welded one to another. For instance, it could be the case that the interface transmits normal displacements and tractions, but not tangential ones. If the tangential tractions vanish for both solids at the interface, while the tangential displacement suffers a jump, then the boundary conditions resemble those found on the faces of an ideal crack.

For this loose bond, the displacement/traction vectors are thus

$$\boldsymbol{\eta}(0) = \begin{bmatrix} U_1(0) \\ U_2(0) \\ 0 \\ T_{22}(0) \end{bmatrix}, \qquad \hat{\boldsymbol{\eta}}(0) = \begin{bmatrix} \hat{U}_1(0) \\ U_2(0) \\ 0 \\ T_{22}(0) \end{bmatrix}, \tag{4.5}$$

where $|U_1(0) - \hat{U}_1(0)| \neq 0$ is the displacement jump. These result in the following determinantal equation for the constants $c_1$, $c_2$, $\hat{c}_1$, $\hat{c}_2$:

$$\begin{vmatrix} 1 & \alpha_L & 1 & \hat{\alpha}_L \\ 1+\alpha_T^2 & 2\alpha_L & 0 & 0 \\ 0 & 0 & (1+\hat{\alpha}_T^2) & 2\hat{\alpha}_L \\ -2\mu\alpha_T & -\mu\,(1+\alpha_T^2) & 2\hat{\mu}\hat{\alpha}_T & \hat{\mu}\,(1+\hat{\alpha}_T^2) \end{vmatrix} = 0. \tag{4.6}$$

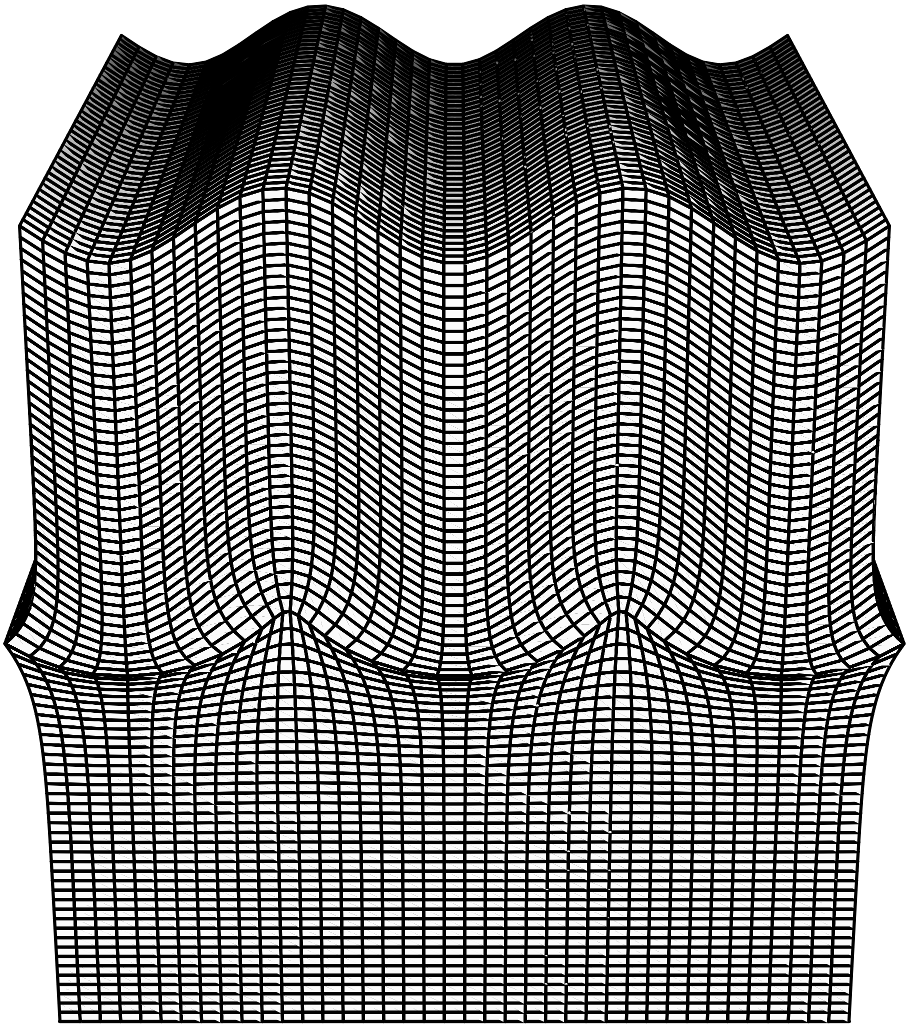

Figure 10: Displacements due to a Stoneley wave at the interface between Nickel (top) and Silica (bottom). Its amplitude decays exponentially with distance from the interface. The Stoneley wave speed (2921 m/s) is very close to the speed of the bulk shear wave in Nickel (2922 m/s), which is why the decay occurs over much greater distances in the upper half-space than in the bottom one.

Expanding the determinant gives

$$\frac{\rho v_T^4}{\sqrt{1-\dfrac{v^2}{v_L^2}}}\left[\left(2-\frac{v^2}{v_T^2}\right)^2-4\sqrt{1-\frac{v^2}{v_T^2}}\sqrt{1-\frac{v^2}{v_L^2}}\right] + \frac{\hat{\rho}\hat{v}_T^4}{\sqrt{1-\dfrac{v^2}{\hat{v}_L^2}}}\left[\left(2-\frac{v^2}{\hat{v}_T^2}\right)^2-4\sqrt{1-\frac{v^2}{\hat{v}_T^2}}\sqrt{1-\frac{v^2}{\hat{v}_L^2}}\right]=0, \quad (4.7)$$

the *slip wave equation.* Like the Rayleigh and Stoneley wave equations, it is non-dispersive. Like the Stoneley wave equation, it may or may not have a solution, depending on the values of the material parameters, see examples below the diagonal in Table 2.

It is clear from equation (4.7) that if the two half-spaces are filled with the same material, then the slip wave travels at the Rayleigh wave speed, because (4.7) then coincides with (3.23). It can be shown that when it exists, a slip wave travels at a speed which is intermediate between the Rayleigh wave speeds of the two solids, see in Table 2 the computed values of slip wave speeds below the diagonal, compared to the Rayleigh wave speeds in the diagonal. It can also be shown that for two isotropic solids in loose contact, at most one slip wave may exist, whereas some combinations of *anisotropic* solids allow for the existence of a second, faster, slip wave.

Table 2. *Speeds of localized elastic waves at the plane interface of 2 isotropic solids, for the pairings of 5 different linear isotropic elastic solids* (m/s). *Above the diagonal: Stoneley waves; Below the diagonal: Slip waves. A cross indicates that no localized interface wave exist for the pairing. On the diagonal, the speed is that of the slip wave when the two sides are made of the same material; it is the same as the Rayleigh wave speed for that material.*

| | Silica | Aluminum | Iron | Steel | Nickel |
|---|---|---|---|---|---|
| Silica | 3400 | X | 3196 | 3175 | 2921 |
| Aluminum | 2995 | 2849 | X | X | 2891 |
| Iron | 3033 | X | 2966 | X | X |
| Steel | X | X | X | 2931 | X |
| Nickel | 2812 | X | X | X | 2738 |

## 4.3 Scholte waves

Now consider the case where one of the half-spaces is filled with a perfect fluid. This set-up might for instance be appropriate to describe the propagation of seismic waves at the bottom of the oceans. It was first elaborated by Scholte (1948).

Here the normal component of the displacement and traction is perfectly transmitted through the interface, while the shear stress must vanish there. It follows that the speed wave equation can be deduced from (4.7), simply be replacing $\hat{\rho}$, $\hat{v}_T$, $\hat{v}_L$ with $\rho_f$ (the fluid mass density), 0 (shear waves are not possible in an ideal fluid), and $c$ (the speed of sound in the fluid), respectively. It reduces to

$$\frac{\rho v_T^4}{\sqrt{1-\dfrac{v^2}{v_L^2}}}\left[\left(2-\frac{v^2}{v_T^2}\right)^2-4\sqrt{1-\frac{v^2}{v_T^2}}\sqrt{1-\frac{v^2}{v_L^2}}\right]+\frac{\rho_f v^4}{\sqrt{1-\dfrac{v^2}{c^2}}}=0, \qquad (4.8)$$

the *Scholte wave equation.* Like the Rayleigh, Stoneley, and slip waves, the Scholte wave is non-dispersive. Like the Rayleigh wave, and unlike the Stoneley and slip waves, it always exist, for all values of material parameters.

Take for instance water as the fluid, with $\rho_f = 1.0 \times 10^3$ kg/m$^3$ and $c = 1.5 \times 10^3$ m/s. Then the Scholte wave at a Water/Aluminum interface travels at 1496 m/s. For the other solids in Table 1, the Scholte wave travels at a speed which is even closer to the speed of sound (within 1%). However it is not a general rule that the Scholte wave speed should be close to the speed of sound in the fluid. Hence, consider a Water/Ice interface (for Ice, $\rho = 910$ kg/m$^3$, $\lambda = 7.4$ GPa, $\mu = 3.6$ GPa): there the Scholte wave travels at 1305 m/s.

## 4.4 Interface waves in anisotropic solids

For anisotropic solids, it is possible to show that a Stoneley wave may or may not exist, depending on material properties; that when it exists, it is unique and propagates at a speed which is larger than that of the slowest Rayleigh wave associated with the two solids. This situation is similar to the case where both solids are isotropic.

In contrast to the isotropic case, there may be zero, one, or *two* slip waves for anisotropic solids, and there exists always one but sometimes *two* Scholte waves.

## 5 Concluding remarks

This short chapter only had room for some basic solutions of the equations of linear elastodynamics. The topic of wave propagation in unbounded and bounded media is quite broad and many aspects have been completely ignored: wave motion with axial symmetry, motion in finite waveguides, diffraction and refraction of waves, impact stress waves and transient waves, point source stimulation, ....

All the waves and concepts exposed in this chapter are related to those topics. Elastic waves are used extensively in many aspects of science and of everyday life such as: the modeling of seismic waves, the non-destructive evaluation of surfaces and cracks, the acoustic determination of elastic constants and of the stiffness of biological soft tissues in medical imaging, to name but a few examples. We hope that our choice has given an indication of the vast the complexity and beauty of the problems encountered in elastodynamics.

# Glossary

- **Progressive plane waves**: Waves such that the initial profile at $t = 0$ is translated along one direction at a given speed.
- **Linearly polarized waves**: Waves with amplitude proportional to a real vector. If this vector is parallel to the direction of propagation, then the wave is longitudinal; if it is orthogonal to that direction, then the wave is transverse.
- **Homogeneous/Inhomogeneous waves**: Waves with planes of constant phase parallel/not parallel to planes of constant amplitude.
- **Bulk waves**: Waves propagating in unbounded media.
- **Surface waves**: Waves leaving the boundary of a semi-infinite solid free of traction.
- **Rayleigh wave**: Inhomogeneous surface wave propagating in a homogeneous semi-infinite solid, with amplitude decaying away from the surface.
- **Stoneley waves**: Inhomogeneous wave propagating at the interface of two homogeneous semi-infinite solids, when the bond is rigid (perfect transmission of displacement and traction), with amplitude decaying away from the interface.
- **Slip waves**: Inhomogeneous wave propagating at the interface of two homogeneous semi-infinite solids, when the bond is slippery (no tangential traction there, but a jump in the tangential displacement), with amplitude decaying away from the interface.
- **Scholte waves**: Inhomogeneous wave propagating at the interface of half-spaces, one filled with a homogeneous semi-infinite solid, the other with a perfect fluid, with amplitude decaying away from the interface.
- **Love waves**: Linearly polarized transverse surface wave propagating on a layer rigidly bonded onto a semi-infinite substrate, with amplitude decaying away from the layer.

# Nomenclature

- $\boldsymbol{n}$ unit vector of the direction of propagation.
- $\boldsymbol{A}$ amplitude vector.
- $\omega$ real angular frequency.
- $v$ speed.
- $\{\bullet\}^+$ real part of a complex quantity.
- $\boldsymbol{k} = k\boldsymbol{n}$ wave vector.
- $\boldsymbol{s} = v^{-1}\boldsymbol{n}$ real slowness vector.
- $\boldsymbol{S}$ complex slowness vector.
- $\boldsymbol{u}$ displacement vector.
- $\epsilon_{ij}$ strain tensor.
- $\sigma_{ij}$ stress tensor.
- $c_{ijhk}$ elastic stiffnesses.
- $\lambda$, $\mu$ Lamé coefficients.
- $\nu$ Poisson ratio.
- $\boldsymbol{Q}(\boldsymbol{n})$ acoustical tensor.

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