can be found using partial fractions.

For a given population it is estimated that $k = 0.029$ and $\ell = 2.941 \times 10^{-12}$. What will the size of this population tend to in the long term, according to the Logistic Law?

9.

10.

10.10 Logic

1. [Truth tables]
   For each of the Boolean functions

   (i) $f(x, y) = x \cdot y \quad mod\ 2$

   (ii) $f(x, y) = x + y \quad mod\ 2$

   (iii) $f(x, y) = (x + y) + x \cdot y \quad mod\ 2$

   complete the following truth table.

x	y	$f(x, y)$
1	1	
1	0	
0	1	
0	0	

2. [Truth tables]
   Define

   $$\overline{x} = 1 + x \quad mod\ 2.$$

   Write out the truth table for each of the Boolean functions

   (i) $f(x, y) = \overline{x \cdot y} \quad mod\ 2,$

   (ii) $f(x, y) = \overline{x \cdot \overline{y}} \quad mod\ 2.$

3. [Truth tables]
   Write out the truth tables for the following Boolean functions. In each case decide if the function is a tautology, a contradiction or neither.

   (i) $((A \Rightarrow B) \Rightarrow B) \Rightarrow B$,

   (ii) $(A \Rightarrow B) \vee (B \Rightarrow A)$,

   (iii) $(\neg A) \Rightarrow (A \wedge B)$,

   (iv) $(A \Rightarrow B) \Leftrightarrow ((\neg A) \vee B)$.

4. [Truth tables]
   Write out the truth tables for the following Boolean functions. In each case decide if the function is a tautology, a contradiction or neither.

   (i) $(A \Leftrightarrow ((\neg B) \vee C)) \Rightarrow ((\neg A) \Rightarrow B))$,

   (ii) $(A \wedge B) \Rightarrow (A \vee C)$,

   (iii) $(A \Rightarrow (B \vee C)) \vee (A \Rightarrow B)$.

5. [Truth tables]

   (i) Decide whether or not $(\neg A) \vee B$ is logically equivalent to $(\neg B) \vee A$.

   (ii) Decide whether or not $\neg(A \Leftrightarrow B)$ is logically equivalent to $A \Leftrightarrow (\neg B)$.

6. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

   > If Murphy is a Communist, Murphy is an atheist. Murphy is an atheist. Hence, Murphy is a Communist.

7. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

   > If the temperature and air pressure remained constant, there was no rain. The temperature did remain constant. Therefore, if there was rain, then the air pressure did not remain constant.

8. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

   > If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security. If other countries feel endangered, they may start a preventative war. If our people will get a false sense of security, they will put less effort into preserving peace. If fallout shelters are not built, we run the risk of tremendous losses in the event of war. Hence, either other countries may start a preventative war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war.

9. [Logical validity]
   Represent the following argument by a truth function. Then decide if the argument is logically valid.

If capital investment remains constant, then government spending will increase or unemployment will result. If government spending will not increase, taxes can be reduced. If taxes can be reduced and capital investment remains constant, then unemployment will not result. Hence government spending will increase.

10. [Logical validity]
Give a formula, in terms of the connectives $\neg$, $\wedge$ and $\vee$, for the truth function $f(x, y)$ defined by the following truth table.

x	y	$f(x,y)$
T	T	F
F	T	T
T	F	T
F	F	T

10.11 Complex numbers

1. [Basic arithmetic]
For $w = 5 + 5\mathbf{i}$ and $z = 3 - 4\mathbf{i}$ express the following complex numbers in the for $x + y\mathbf{i}$:

$$w + z, \quad w - z, \quad wz, \quad \frac{w}{z}\ .$$

2. [Argument and modulus]
Find the argument $Arg(z)$ and modulus $|z|$ of the following complex numbers.

(i) $z = 2 + 2\sqrt{3}\mathbf{i}$

(ii) $z = -5 + 5\mathbf{i}$

(iii) $z = \dfrac{3\mathbf{i}^{30} - \mathbf{i}^{19}}{2\mathbf{i} - 1}$

(iv) $z = \dfrac{5 + 5\mathbf{i}}{3 - 4\mathbf{i}} + \dfrac{20}{4 + 3\mathbf{i}}$

3. [De Moivre's Theorem]
Express each of the following numbers z in the form $x + y\mathbf{i}$.

(i) $z = vw$ where $v = 3(\cos 40° + \mathbf{i}\sin 40°)$ and $w = 4(\cos 80° + \mathbf{i}\sin 80°)$.

(ii) $z = v^7/w^3$ where $v = 2(\cos 15° + \mathbf{i}\sin 15°)$ and $w = 4(\cos 45° + \mathbf{i}\sin 45°)$.

(iii) $z = \left(\dfrac{1 + \sqrt{3}\mathbf{i}}{1 - \sqrt{3}\mathbf{i}}\right)^{10}.$

4. [Euler's formula]
   We define
$$e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$$
   and, more generally,
$$e^{x+\mathbf{i}y} = e^x e^{\mathbf{i}y} = e^x(\cos\theta + \mathbf{i}\sin\theta)\ .$$
   Use this definition to prove the following identities.
   (i) $\cos\theta = \dfrac{e^{\mathbf{i}\theta} + e^{-\mathbf{i}\theta}}{2}$.
   (ii) $\sin\theta = \dfrac{e^{\mathbf{i}\theta} - e^{-\mathbf{i}\theta}}{2\mathbf{i}}$.
   (iii) $\cos^2\theta + \sin^2\theta = 1$.

5. [Square roots]
   Find the square roots of $-15 - 8\mathbf{i}$.

6. [Complex roots of unity]
   List all cube roots of 1. Hence factorize the polynomial $x^3 + 1$ as a product of (complex) linear polynomials.

7. [Complex roots of unity]
   Use De Moivre's Theorem to express $\cos 3\theta$ as a sum of powers of $\cos\theta$. Similarly, express $\sin 3\theta$ as a sum of powers of $\sin\theta$.

8. [Complex roots]
   Factorize $x^5 + x^4 + x^3 + x^2 + x + 1$ as a product of real linear and quadratic factors.

9. [Complex roots of unity]
   Deduce that
$$\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + \sin\frac{4\pi}{3} + \sin\frac{5\pi}{3} = 0$$
   from the fact that the nth roots of unity sum to zero.

10. [Complex roots]
   Find all complex numbers z that satisfy $z^5 = -32$. Hence factorize the polynomial
$$x^5 + 32$$
   as a product of real polynomials of degree at most 2.

10.12 Systems of Equations

1. [System of n equations in n unkowns]
A brewery produces Brown Ale, Dark Brown Ale and Porter. The following table summarizes the amount of malt, hops and yeast used to produce one keg of beer together with the total amount of these resources available per day.

Resource	Brown Ale	Dark Brown Ale	Porter	Daily available
Yeast	3 kg	2 kg	6 kg	250 kg
Hops	4 kg	3 kg	4 kg	220 kg
Malt	4 kg	2 kg	3 kg	170 kg

   (a) Let x, y, z be the number of kegs of Ale, Brown Ale and Porter produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.

   (b) Find the values of x,y,z which ensure that the daily supply of hops, malt and yeast are fully used.

2. [System of n equations in n unkowns]
A small dairy produces three cheeses: mild, standard and mature. The following table summarizes the amount of energy, milk and labour used to produce one box of each of the three cheeses together with the amount of these resources available per day.

Resource	Mild A	Standard B	Mature C	Daily available
Energy	2 kWh	3 kWh	2 kWh	100 kWh
Milk	4 L	4 L	3 L	150 L
Labour	3 h	4 h	6 h	170 h

   (a) Let x, y, z be the number of boxes of mild, standard and mature cheese produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.

   (b) Find the values of x,y,z which ensure that all resources are fully used.

3. [System of n equations in m unkowns]
Find one solution P to the following system of linear equations.

$$\begin{array}{rcrcrcr} 3x & + & 5y & + & 7z & = & 15 \\ x & + & y & + & z & = & 1 \end{array}$$

Then find a vector V such that all solutions are of the form $P + \lambda V$ for some scalar $\lambda \in \mathbb{R}$.

4. [System of n equations in m unkowns]
Find one solution P to the following system of linear equations.

$$\begin{array}{rcrcrcrcr} w & + & 3x & + & 3y & + & 2z & = & 1 \\ 2w & + & 6x & + & 9y & + & 5z & = & 5 \\ -w & + & -3x & + & 3y & + & & = & 5 \end{array}$$

Then find vector V, V' such that all solutions are of the form $P + \lambda V + \mu V'$ for scalars $\lambda, \mu \in \mathbb{R}$.

5. [Inconsistent systems]
   Find all values of k for which the following system has no solutions.

$$\begin{array}{rcrcl} x & + & ky & = & 0 \\ kx & + & 9y & = & 1 \end{array}$$

6. [Inverse matices]
   Use row operations to find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 4 \\ 6 & 1 & 7 \end{pmatrix}.$$

7. [Inverse matices]
   Use row operations to find the inverse of

$$A = \begin{pmatrix} 3 & 2 & 6 \\ 4 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix}.$$

   (Compare Question 1.)

8. [Inverse matices]
   Use row operations to find the inverse of

$$A = \begin{pmatrix} 3 & 4 & 4 \\ 2 & 3 & 2 \\ 6 & 4 & 3 \end{pmatrix}.$$

   (Compare Question 7.)

9.

10.