Integration

Chapter 5 of [Stewart] contains background and examples related to the following integration problems.

1. [Integrals as areas]
   Evaluate the following integrals.

   (i) $\displaystyle\int_{-2}^{3} x+1\,dx$

   (ii) $\displaystyle\int_{-2}^{3} |x+1|\,dx$

   The *absolute value function* is defined as $|x| = x$ for $x \geq 0$, $|x| = -x$ for $x < 0$.

2. [Integrals as areas]
   Evaluate the following integrals.

   (i) $\displaystyle\int_{-2}^{3} \lfloor x+1 \rfloor\,dx$

   (ii) $\displaystyle\int_{-2}^{3} \lfloor x+1 \rfloor^2, dx$

   (iii) $\displaystyle\int_{-2}^{3} \lceil x+1 \rceil\,dx$

   (iv) $\displaystyle\int_{-2}^{3} \lceil x+1 \rceil^2, dx$

   The *floor function* is defined as $\lfloor x \rfloor = n$ where n is the largest integer $n \leq x$.

   The *ceiling function* is defined as $\lceil x \rceil = n$ where n is the smallest integer $n \geq x$.

3. [Integral of sums and scalar products]
   Evaluate the following integrals.

   (i) $\displaystyle\int_{-3}^{2} x+1+|x+1|, dx$

   (ii) $\displaystyle\int_{-3}^{2} \tfrac{1}{4}(x+1+\lfloor x+1 \rfloor), dx$

4. [Areas of bounded regions]
   Calculate the areas of the regions bounded by:

   (i) x-axis and $y = x^4 + x^2 + 1$ between $x = -1$ and $x = 1$.

   (ii) x-axis and $y = x^2 + x - 2$.

(iii) $y = x + 1$ and $y = x^2 + 2$ between $x = -1$ and $x = 1$.

(iv) $y = x^2$ and $y = 2 - x$.

5. [Fundamental Theorem of Calculus]
A particle is shot straight upwards from the ground with initial velocity $98m/s$. Its velocity after t seconds is $v = -9.8t + 98$. At what time t does it reach its maximum height? What maximum height is achieved?

6. [Fundamental Theorem of Calculus]
The rate of flow of water into an initially empty tank is $100 - 3t$ galllons per minute at time t minutes. How much water flows into the tank during the interval from $t = 10$ to $t = 20$ minutes?

7. [Fundamental Theorem of Calculus]
The birth rate in a certain city t years after 1960 was $13 + t$ thousands of births per year. Set up and evaluate an appropriate integral to compute the total number of births that occurred between 1960 and 1980.

8. [Fundamental Theorem of Calculus]
The city in the previous problem had a death rate of $5 + t/2$ thousands per year t years after 1960. If the population of the city was 125 000 in 1960, what was its population in 1980? (Consider both births and deaths.)

9. [Fundamental Theorem of Calculus]
On the moon the acceleration due to gravity is $1.6m/sec^2$. If a rock is dropped into a crevasse, how fast will it be going just before it hits the bottom $30\,sec$ later?

10. [Fundamental Theorem of Calculus]
A heavy object is dropped from the top of the Eiffel Tower. The tower is 324 metres high. Approximately how long will the object take to reach the ground? (Acceleration due to gravity is $g = 9.8m/sec^2$.)

Techniques of Integration

Chapter 7 of [Stewart] contains background and examples related to the following problems on indefinite integrals. (An indefinite integral is the same thing as an anti-derivative.)

1. [Algebraic simplification]
Determine the following integrals:

(i) $\displaystyle\int (x^2 - 1)(x + 1)\, dx$

(ii) $\displaystyle\int (x^3 + 1)^2\, dx$

2. [Simple substitution]
   Use a substitution to determine the followings integrals.

   (i) $\displaystyle\int x^3 \cos(x^4 + 2)\,dx$

   (ii) $\displaystyle\int \sqrt{2x + 1}\,dx$

   (iii) $\displaystyle\int \frac{x}{\sqrt{1 - 4x^2}}\,dx$

3. [Logarithms]
   Determine the following integrals.

   (i) $\displaystyle\int e^{2x}\,dx$

   (ii) $\displaystyle\int \frac{1}{x}\,dx$

   (iii) $\displaystyle\int \frac{2x}{x^2 + 8}\,dx$

   (iii) $\displaystyle\int \frac{x^2 + x}{x^3 + 3x + 8}\,dx$

   (iv) $\displaystyle\int \frac{6x^2 + 4x + 2}{x^3 + x^2 + x + 8}\,dx$

4. [Integration by parts]
   Use integration by parts to determine the following integrals:

   (i) $\displaystyle\int x \sin(x)\,dx$

   (ii) $\displaystyle\int t^2 e^t\,dt$

   (iii) $\displaystyle\int e^x \sin(x)\,dx$

5. [Reduction formulae]
   Prove the reduction formula

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

   where $n \geq 2$ is an integer.

6. [Trigonometric substitutions]
   Evaluate
   $$\int \frac{\sqrt{16-x^2}}{x^2}\,dx\ .$$
   (Hint: Try $x = a\sin\theta$ for a suitable value of a.)

7. [Trigonometric substitutions]
   Evaluate
   $$\int \frac{x^3}{\sqrt{x^2+9}}\,dx\ .$$
   (Hint: Try $x = a\tan\theta$ for a suitable value of a.)

8. [Partial fractions]
   Evaluate
   $$\int \frac{x+5}{x^2+x-2}\,dx\ .$$

9. [Partial fractions]
   Evaluate
   $$\int \frac{1}{x^3-x^2-x+1}\,dx\ .$$

10. [Partial fractions]
   Evaluate
   $$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx\ .$$
   (Hint: you'll first need to apply long division.)

Logarithms and differential equations

1. [Verifying solutions]
   Show that $y = Ce^{3x} - e^{2x}$ is a solution of the differential equation
   $$\frac{dy}{dx} - 3y = e^{2x}$$
   for any constant C. Determine the value of C that ensures $y(0) = 3$.

2. [Verifying solutions]
   Show that $y = Ae^{kt}$ is a solution of the differential equation
   $$\frac{dy}{dt} = ky$$
   for any constant A. Determine the values of the constants A and k that ensure $y(0) = 60$ and $y(5) = 30$.

3. [Applications of differential equations]
A cup of coffee in a room at $20°$C cools for $80°$C to $50°$C in 5 minutes. How long will it take to cool to $40°$C? (Newton's Law states that a hot object cools at a rate proportional to the excess of its temperature above room temperature.)

4. [Applications of differential equations]
The Malthusian Law states that the size $y(t)$ of a population at time t is governed by the differential equation
$$\frac{dy}{dt} = ky \ .$$
In other words, the rate of change of a population is proportional to the size of the population.

   The population of the world in 1960 was 3.06 billion. Use the Malthusian Law with $k = 0.02$ to estimate the population in the year 2016.

5. [Separable differential equations]
Find the solution to the differential equation
$$y^2 \frac{dy}{dt} = t^2, \quad y(0) = 27 \ .$$

6. [Separable differential equations]
Solve the differential equation
$$e^y \frac{dy}{dt} - t - t^3 = 0, \quad y(0) = 1 \ .$$

7. [Separable differential equations]
Solve the differential equation
$$\frac{dy}{dt} = 1 + y^2, \ y(0) = 0 \ .$$

8. [Applications of differential equations]
The Logistic Law states that the size $y(t)$ of a population at time t is governed by the differential equation
$$\frac{dy}{dt} = ky = \ell y^2 \ .$$

For a given population it is estimated that $k = 0.029$ and $\ell = 2.941 \times 10^{-12}$. What will the size of this population tend to in the long term, according to the Logistic Law?