MA302 Complex Variable, Problem Sheet 1

1. Express the following in the form $x + iy$ and draw them in the comlex plane.

   (a) $\dfrac{2-3i}{4-i}$; (b) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$; (c) $\dfrac{5}{(1-i)(2-i)(3-i)}$; (d) $(1+i)^{-2}$.

2. Show that, for all complex numbers z, z_1, z_2:

   (a) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$; (b) $\overline{-z} = -\overline{z}$; (c) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$; (d) $\overline{z^{-1}} = \overline{z}^{-1}$;
   (e) $\mathrm{Re}(z) = \frac{1}{2}(z+\overline{z})$; (f) $\mathrm{Im}(z) = \frac{1}{2i}(z-\overline{z})$; (g) $\overline{\overline{z}} = z$.

3. Show that, for all z, z_1, $z_2 \in \mathbb{C}$ (a) $|z_1 z_2| = |z_1||z_2|$; (b) $|z^{-1}| = |z|^{-1}$; (c) $|\overline{z}| = |z|$.

4. Express each of the following in polar form and draw them in the complex plane.

   (a) $2-2i$; (b) $-1+\sqrt{3}\,i$; (c) $\sqrt{2}\,i$; (d) $\dfrac{-2}{1+\sqrt{3}\,i}$.

5. (a) Let n be an integer. Show that if z is a complex number with $z^n = 1$, then $\overline{z}^n = 1$ as well.

   (b) Find *all* complex numbers z satisfying $z^3 = 1$ and draw them in the complex plane.

6. Prove the Triangle Inequality: $|z_1 + z_2| \le |z_1| + |z_2|$ for all z_1, $z_2 \in \mathbb{C}$. [Hint. $|z_1+z_2|^2 = z_1\overline{z_1} + 2Re(z_1\overline{z_2}) + z_2\overline{z_2} \le ||z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$.]

MA302 Complex Variable, Problem Sheet 2

1 Find all possible values of the following roots and locate them graphically.
(a) $(-1-\sqrt{3}\,i)^{1/2}$, (b) $(-4+4i)^{1/5}$, (c) $(-8)^{1/3}$, (d) $(-11-2i)^{1/3}$.

2 Solve the equations

(a) $z^4 + 81 = 0$;

(b) $z^6 + 1 = \sqrt{3}\,i$.

3 Find the images of the following curves under the mappings $f(z) = z^3$ and $g(z) = \frac{1}{z}$.

(a) Rays emanating from the origin.

(b) Circles centred at the origin.
Your answer should be like "the image of the ray emanating from the origin at an angle θ with the x-axis is ... because ..." and similarly for circles.

4 Find the following limits. (a) $\lim_{z\to 2i}(i\,z^4 + 3\,z^2 - 10i)$

(b) $\lim_{z\to 2i}\left(\frac{z^2+4}{z-2i}\right)$

(c) $\lim_{z\to 1+i}\left(\frac{z-1-i}{z^2-2\,z+2}\right)^2$

5 Describe the following sets in words and sketch them.
(a) $\{z \in \mathbb{C} \mid 3 = |z-2+i|\}$ (b) $\{i + 2e^{i\theta} \mid 0 \le \theta < 2\pi\}$
(c) $\{t^2 + it \mid 0 \le t \le 2\}$

6 Find $f'(z)$ in the following cases.
(a) $f(z) = 3z^2 - 2z + 4$ (b) $f(z) = (1-4z^2)^3$ (c) $f(z) = \frac{z-1}{2z+1}$ $(z \neq \frac{1}{2})$

MA302 Complex Variable, Problem Sheet 3

1. Express the following complex numbers in the form $x+iy$, $x, y \in \mathrm{Re}$.
(a) $\dfrac{1+2i}{3-4i} - \dfrac{4-3i}{2-i}$ (b) $e^{2+\pi i/2}$ (c) $\overline{(2-3i)}(2+i)$ (d) $[\sqrt{2}(1-i)]^7$

2. Let $f(z) = \exp(z) = e^z$. Find the images under f of (a) the vertical line where $\mathrm{Re}(z) = a$ and (a) the horizontal line where $\mathrm{Im}(z) = b$, where $a, b \in \mathrm{Re}$ are constants.

3. For each of the following functions show that $f'(z)$ does not exist at any point away from zero.

   (a) $f(z) = f(x,y) = 2x + i\,xy^2$ (b) $f(z) = z - \overline{z}$ (c) $f(z) = z^2\overline{z}$

   (d) $f(z) = f(x,y) = e^x e^{-iy}$

4. For the following functions determine where $f'(z)$ exists and find its value. Here $z = x + yi$.

   (a) $f(z) = x^2 + i\,(y^2 - 4y)$ (b) $f(z) = x^2 + i\,y^2$ (c) $f(z) = z\,\mathrm{Im}(z)$

5. Suppose $f(z) = f(x,y) = u(x,y) + iv(x,y)$ is complex differentiable, where u and v are real valued functions. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

MA302 Complex Variable, Problem Sheet 4

1. Compute the following.
   (a) $\log(3+2i)$ (b) $\log(1-i)$ (c) $\sin(i)$ (d) $\cos(\pi i)$

2. Find all solutions to the following equations.
   (a) $e^{2z}-2i=0$ (b) $\sin(z)+2i=0$ (c) $(z-i)^5=1$ (d) $z^3=1+i$

3. Verify the following identities.
   (a) $\sin^{-1}(z)=-i\log(iz+(1-z^2)^{1/2})$ (b) $\tan^{-1}(z)=\frac{i}{2}\log\frac{i+z}{i-z}$

4. For each of the following real valued functions of two real variables decide whether they are harmonic and if so find a harmonic conjugate.
   (a) $u(x,y)=x^2-y^2$ (b) $u(x,y)=xy^3-x^3y$ (c) $u(x,y)=x^3+y^3$

5. Find all points where $f(z)=f(x+iy)=x^3-2x^2-3xy^2+i(3yx^2-y^3+y^2)$ is differentiable.

MA302 Complex Variable, Problem Sheet 5

1. For a real variable x, the hyperbolic sinc and cosine are defined as

   $$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}).$$

   For (i) $f(z) = \sin(z)$ and (ii) $f(z) = \cos(z)$ with $z = x + iy$ find the real part $u(x, y)$ and imaginary part $v(x, y)$ of f as functions of x and y.

2. Decide which curves are defined by the following parameterisations and draw them in the complex plane. Then find parametrisations for the same curves with the opposite orientation.
   (a) $z(t) = 1 + 2\cos(t) + i\sin(t)$, $0 \leq t \leq 2\pi$
   (b) $z(t) = t - 2 + it^2 + 3i$, $0 \leq t \leq 2$

3. Find parametrizations of the following curves.
   (a) The positively (couter clock-wise) oriented circle $|z - 4i| = 2$.
   (b) The line segment from $2i - 1$ to $6 + 3i$.
   (c) The semicircle through $-2i$, 2 and $2i$.

4. Evaluate the following contour integrals.

   $$\int_C (|z|+\bar{z})\, dz, \qquad \int_C (z\mathrm{Re}(z) - \bar{z}\mathrm{Im}(z))\, dz \quad \text{and} \quad \int_C (z^2 - (2-3i)z + i)\, dz$$

   where C is
   (a) the line segment from -1 to i;
   (b) the quarter circle connectiong -1 to i;
   (c) the path that follows the x-axis from -1 to 0 and from there the y-axis to i.

MA302 Complex Variable, Problem Sheet 6

1. Evaluate the following integrals when C is the positively oriented boundary path of the square with vertices ± 2 and $\pm 2i$.

$$(a) \quad \int\limits_C \frac{z^2 - 4z}{(z-1)^3}\, dz \qquad\qquad (b) \quad \int\limits_C \frac{e^{2z} - z^2}{(1-i)^7}\, dz$$

2. Verify that $\int\limits_a^b |z'(t)|\, dt$ is the lenght of the curve $z : [a, b] \longrightarrow \mathbb{C}$ for

   (a) the straight line segment from z_0 to z_1
   (b) a circle of radius R centered at z_0

3. Evaluate the following contour integrals

$$\int_C (|z| + \overline{z})\, dz, \qquad \int_C (z\overline{z} + z^2)\, dz \quad \text{and} \quad \int_C (z^2 - (2-3i)z + i)\, dz$$

   where C is **(a)** the line segment from -1 to i and **(b)** the quarter circle connecting -1 to i.

4. Suppose $f(z)$ is analytic. Show that if $f'(z) = 0$, then $f(z)$ is constant.

5. Show that if f is analytic in the disc of radius R about 0, then so is $g(z) = \overline{f(\overline{z})}$.

MA302 Complex Variable, Problem Sheet 7

1. Find all values of of

   $(a)\quad \log(1+i) \qquad (b)\quad (1+i)^{2i} \qquad (c)\quad \cos^{-1}(3i) \qquad (d)\quad (3i-3)^{1/6}$

2. Let $f(z) = \dfrac{z-4i}{(z^2+4)(z-i)}$. Allowing only positively oriented simple closed curves C, how many different values can $\int_C f(z)\,dz$ have?

3. Let C be the circle with parametrisation $z(t) = a + e^{it}$, where a is a constant. Evaluate the following integrals depending on a.

   $$(a)\quad \int_C \frac{e^z \sin(z)}{(z-a)^3}\,dz \qquad\qquad (b)\quad \int_C \frac{z^2+iz-5+i}{(z-1)^2}\,dz$$

4. State Cauchy's Integral Formula for Derivatives and use it, if applicable, to evaluate the following integrals, where C is the positively oriented circle of radius 3 centred at 1.

   $$(a)\quad \int_C \frac{4iz^3e^z}{(z-i)^3}\,dz \qquad\qquad (b)\quad \int_C \frac{\sin(z)}{(z+3i)^5}\,dz$$

5. Find the Taylor series for $\sin(z)$ centered at (a) zero and (b) $\frac{\pi}{2}$.

6. Find the Laurent series with centre 0 for $\frac{e^{z^2}}{z^3}$. Where in the complex plane does this series converge.

7. Find the Laurent series of

   $$\frac{1}{(z-4)(z-6)}$$

   (a) with centre $z = 4$ if (i) $0 < |z-4| < 2$ (ii) $|z-4| > 2$,

   (b) with centre $z = 6$ if (i) $0 < |z-6| < 2$ (ii) $|z-6| > 2$.

8. Verify by direct calculation, i.e. without the use of any of Cauchy's Formulæ, that

   $$\int_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$$

9. Find the residues of $f(z) = \dfrac{z^2 - iz}{(z^2+1)(z-2i)^2}$ at all its poles.

10. Find the integral of the function in Q9 around the simple closed curve which encloses (i) $-i$ (ii) $2i$ (iii) i (iv) $-i$ and $2i$.