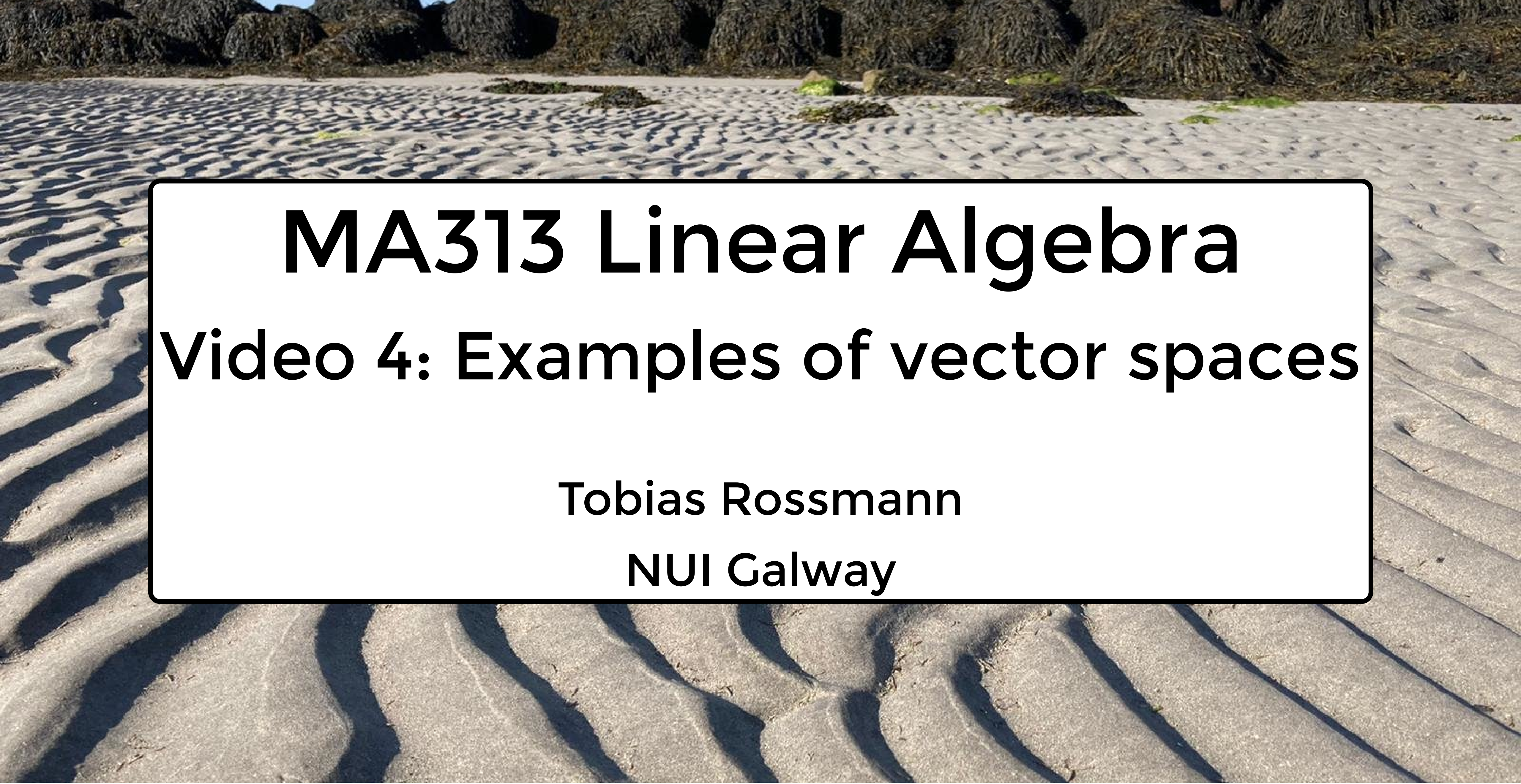


MA313 Linear Algebra

Video 4: Examples of vector spaces

Tobias Rossmann
NUI Galway

Example (matrix spaces)

Let $M_{m\times n}$ be the set of $m \times n$ matrices with real entries. Then $M_{m\times n}$ is a vector space with respect to the following operations:

$$\begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \ddots & \ddots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & \dots & b_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ b_{m1} & \ddots & \ddots & b_{mn} \end{bmatrix} := \begin{bmatrix} a_{11}+b_{11} & \dots & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1}+b_{m1} & \ddots & \ddots & a_{mn}+b_{mn} \end{bmatrix}$$

$$c\begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \ddots & \ddots & a_{mn} \end{bmatrix} := \begin{bmatrix} ca_{11} & \dots & \dots & ca_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ ca_{m1} & \ddots & \ddots & ca_{mn} \end{bmatrix}$$

These are just the usual operations of adding matrices and multiplying a matrix by a scalar! **Exercise:** verify that **V1**-**V8** are really satisfied.

Example (discrete signals)

Let $\mathbb{S}$ be the set of all doubly infinite sequences $(y_k) = (\dots, y_{-1}, y_0, y_1, \dots)$ of real numbers.

Define

$$(y_k) + (z_k) := (y_k + z_k)$$

and

$$c(y_k) := (cy_k),$$

where $(y_k), (z_k) \in \mathbb{S}$ and $c \in \mathbb{R}$. We call the elements of $\mathbb{S}$ **discrete signals**.

Claim: $\mathbb{S}$ (together with the operations defined above) is a vector space.

Why are the elements of $\mathbb{S}$ called discrete signals?

If we measure a continuous signal at discrete times, we obtain an element of $\mathbb{S}$.

More formally, a function f is sent to the discrete signal $(f(k)) = (\dots, f(-1), f(0), f(1), \dots)$.

Note: The vector space $\mathbb{S}$ is "unlike" any $\mathbb{R}^n$.

For instance, there is no finite collection of "coordinates" that can be used to distinguish different elements in all cases. More about this later!

Example (polynomials)

For an integer $n \geqslant 0$, let $\mathbb{P}_n$ consist of all polynomials

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

of degree at most n, where $a_0, \ldots, a_n \in \mathbb{R}$.

We can add polynomials in $\mathbb{P}_n$ in the usual way:

$$(a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_n t^n) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n.$$

We further have

$$cp(t) = ca_0 + ca_1 t + \cdots + ca_n t^n,$$

where $c \in \mathbb{R}$ and $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ as above.

Claim: These operations turn $\mathbb{P}_n$ into a vector space.

Why? Again, this boils down to properties of real numbers!

Example (function spaces)

Let $\mathbb{D}$ be a (completely arbitrary!) set.

Let V be the set of *all* functions $f : \mathbb{D} \to \mathbb{R}$.

Given $f, g \in V$ and $c \in \mathbb{R}$, we define $f + g \in V$ and $cf \in V$ via

$$(f+g)(x) := f(x) + g(x)$$

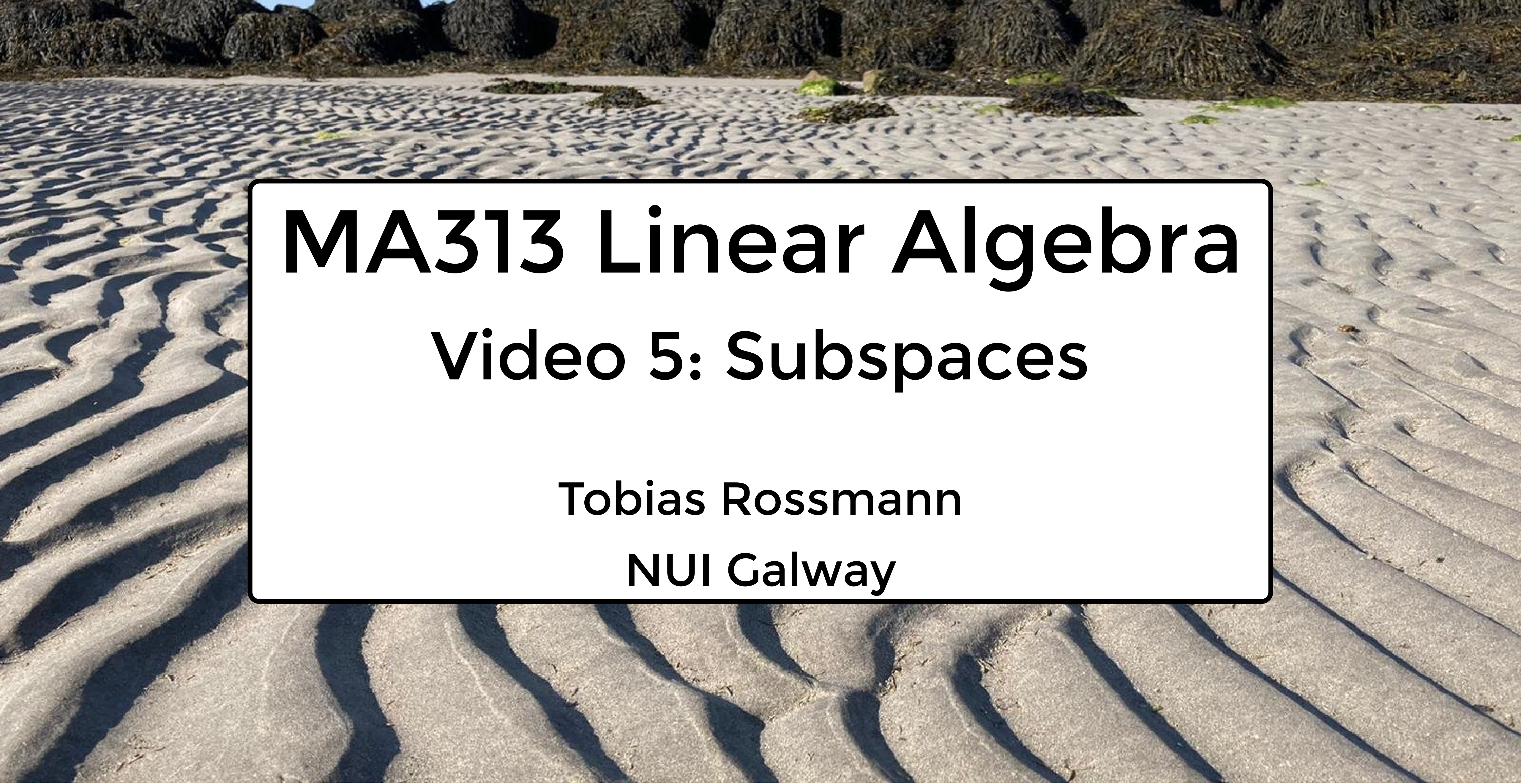
and

$$(cf)(x) := cf(x)$$

for $x \in \mathbb{D}$.

Claim: These operations turn V into a vector space.

Note that calculus "takes place" in this space for $\mathbb{D} = \mathbb{R}, (a, b), \ldots$.



MA313 Linear Algebra

Video 5: Subspaces

Tobias Rossmann

NUI Galway

Recall

A vector space is a collection of "things" that can be added together and multiplied by scalars subject to certain rules.

Many vector spaces in mathematics and its applications don't appear "out of nowhere". Instead, they live within other vector spaces as so-called **subspaces**.

Definition

Let V be a vector space. A **subspace** of V is a subset of V which forms a vector space with respect to the addition and scalar multiplication inherited from V.

Example

The "boring" subspaces of a vector space V are $\{0\}$ and V itself.

How can we recognise subspaces?

Fact

Let $H \subseteq V$ be a subset. Then H is a subspace of V if and only if the following conditions are all satisfied:

- $0 \in H$.
- H is closed under addition in V, i.e. for all $u, v \in H$, we have $u + v \in H$.
- H is closed under multiplication by scalars, i.e. for all $u \in H$ and $c \in \mathbb{R}$, we have $cu \in H$.

Example

Let

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \text{ with } x + y + z = 0 \right\}.$$

Then H is a subspace of $\mathbb{R}^3$.

Example

Let

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \text{ with } x + y + z = 0 \right\}.$$

Then H is a subspace of $\mathbb{R}^3$.

Example

Let

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \text{ with } x + y + z = 0 \right\}.$$

Then H is a subspace of $\mathbb{R}^3$.

Problem (from 2018/2019 exam paper)

Decide (with justification) whether

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x + y \geqslant 0 \right\}$$

is a subspace of $\mathbb{R}^2$.