# MA313 Linear Algebra

## Video 6: Further examples of subspaces

Tobias Rossmann
NUI Galway

## Recall

$\mathbb{P}_n = \{a_0 + a_1 t + \cdots + a_n t^n : a_0, \ldots, a_n \in \mathbb{R}\}$ is the vector space of polynomials of degree at most $n$ in the variable $t$.

## Definition

$$\mathbb{P} := \bigcup_{n=0}^{\infty} \mathbb{P}_n = \{p(t) = a_0 + a_1 t + \cdots + a_n t^n : n \geqslant 0;\ a_0, \ldots, a_n \in \mathbb{R}\}$$

is the vector space of all polynomials in $t$ (without any bounds on the degree!).

Here, the addition and scalar multiplication in $\mathbb{P}$ are just the usual operations: we add and multiply as expected.

## Fact

- $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$ if and only if $m \leqslant n$.
- $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for all $n \geqslant 0$.

Note that

$$\begin{aligned}\mathbb{P}_0 = \mathbb{R} &= \text{constant polynomials} \\ &= \{p(t) \in \mathbb{P} : p'(t) = 0\}\end{aligned}$$

where the **derivative** of $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ is

$$p'(t) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}.$$

More generally,

$$\mathbb{P}_n = \left\{ p(t) \in \mathbb{P} : p^{(n+1)}(t) = 0 \right\}.$$

**Punchline:** the subspaces $\mathbb{P}_n$ of $\mathbb{P}$ can be described as solutions of certain equations!

## Example

Let $\mathbb{D} \subseteq \mathbb{R}$ be a subset. Let $V$ be the vector space of all functions $\mathbb{D} \to \mathbb{R}$ from before.

Recall:

- $(f+g)(x) = f(x) + g(x)$ for $f, g \in V$ and $x \in \mathbb{D}$
- $(cf)(x) = cf(x)$ for $f \in V$ and $c \in \mathbb{R}$

Let

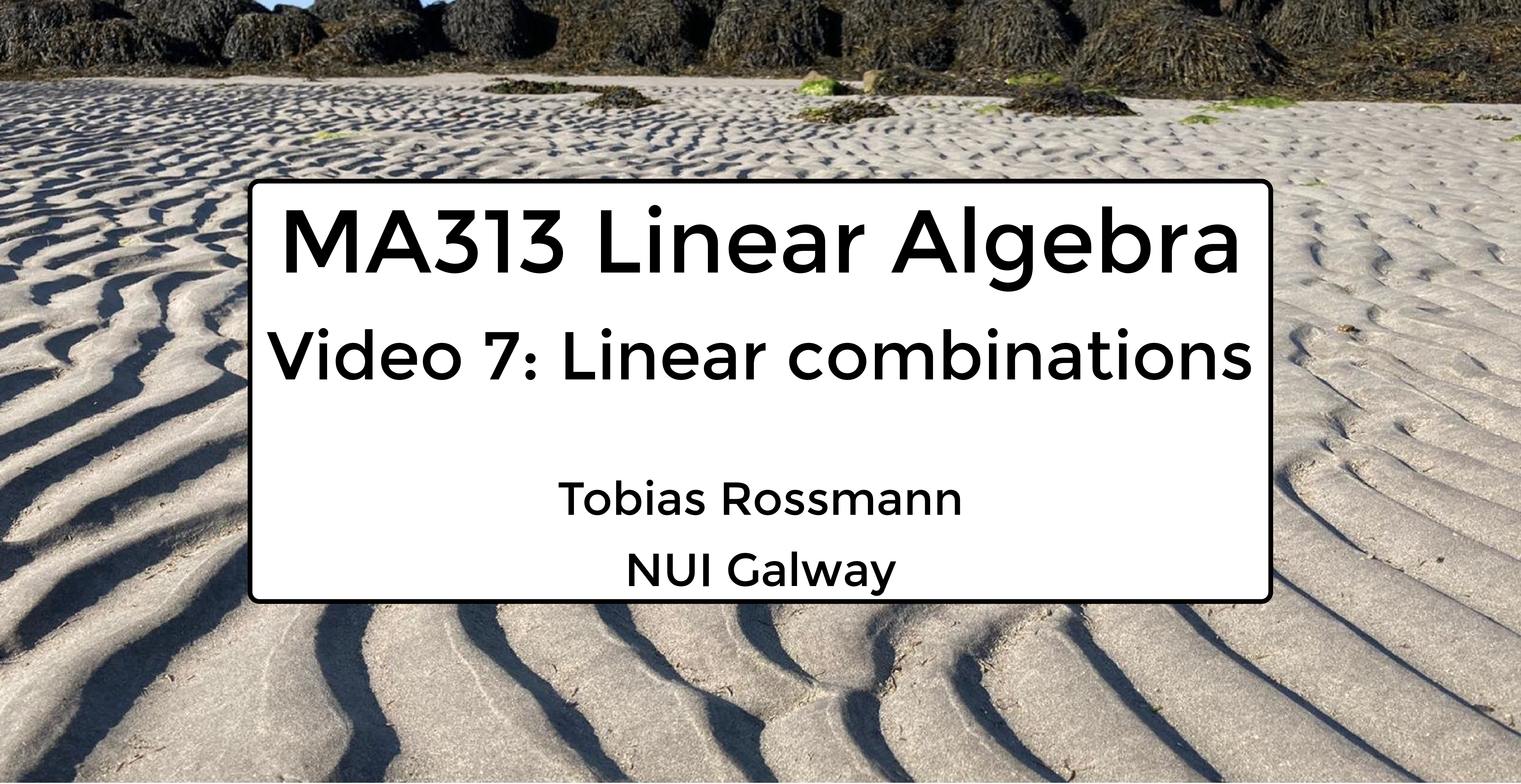
$$C(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{R} : f \text{ is continuous}\} \subseteq V.$$

**Theorem**

$C(\mathbb{D})$ is a subspace of $V$.

Why? Calculus!

- Constant functions are continuous. In particular, the zero vector of $V$ (= the constant function with value zero) belongs to $C(\mathbb{D})$.
- Sums and products of continuous functions are continuous. Hence, $C(\mathbb{D})$ is closed under addition and scalar multiplication (= multiplication by constant functions).



# MA313 Linear Algebra

## Video 7: Linear combinations

Tobias Rossmann
NUI Galway

## Question

How can we describe *all* subspaces of a given vector space?

## Example

There are precisely three types of subspaces of $\mathbb{R}^2$:

- $\{0\}$,
- $\mathbb{R}^2$, and
- lines through the origin.

## Question

How do subspaces arise in "nature"?

- **From the top:**
  all vectors that have "suitable properties"
- **From the bottom:**
  start with some collection of vectors and consider the subspace that they "span"

## Definition

A **linear combination** of vectors $u_1, \ldots, u_p$ in some vector space is a vector of the form

$$c_1 u_1 + \cdots + c_p u_p$$

for scalars $u_1, \ldots, u_p \in \mathbb{R}$.

## Example

In $\mathbb{R}^2$, $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

## Example

Is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -6 \end{bmatrix}$ in $\mathbb{R}^2$?