MA313 Linear Algebra

Video 8: Spans

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Definition

Given vectors $u_1, \ldots, u_p$ in some vector space V, their **span** is

$$\operatorname{span}\{u_1, \ldots, u_p\} := \{c_1 u_1 + \cdots + c_p u_p : c_1, \ldots, c_p \in \mathbb{R}\}.$$

In other words, $\operatorname{span}\{u_1, \ldots, u_p\}$ is the set of all linear combinations of $u_1, \ldots, u_p$ within V.

Theorem

$\text{span}\{u_1, \ldots, u_p\}$ is a subspace of V.

In fact, one can show that $\text{span}\{u_1, \ldots, u_p\}$ is the "smallest" subspace of V which contains each of $u_1, \ldots, u_p$.

Immediate consequences

- Every choice of vectors $u_1, \ldots, u_p$ provides us with an example of a subspace of V.

  (However, *different* sequences of vectors may well span the *same* subspace!)

- If we recognise a *subset* of V as a span, then we know it has to be a subspace!

Example

Let

$$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Claim: H is a subspace of $\mathbb{R}^4$.

Problem (from 2018/2019 exam paper)

Find vectors $u, v, w \in V$ with $V = \mathrm{span}\{u, v, w\}$, where V is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$.

Non-example

Let

$$H = \left\{ \begin{bmatrix} 3s \\ 2+5s \end{bmatrix} : s \in \mathbb{R} \right\}.$$

Is this a subspace of $\mathbb{R}^2$?

Question

Is every subspace the span of some (collection of) vectors?

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Video 9: Null spaces

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Two main sources of subspaces

- Spans of vectors ("from the bottom").
- **Kernels** and **null spaces** of **linear transformations** ("from the top").

  These objects generalise sets of solutions to homogeneous systems of linear equations.

Definition

Let A be an $m \times n$ matrix. The **null space** of A is

$$\operatorname{Nul} A = \{x \in \mathbb{R}^n : Ax = 0\}\,.$$

Recall: for a matrix $A = [a_1 \cdots a_n]$ with columns $a_1, \ldots, a_n \in \mathbb{R}^m$ and a vector $x \in \mathbb{R}^n$, we have

$$Ax = x_1 a_1 + \cdots + x_n a_n.$$

Example

Let

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}.$$

Then

$$A \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and hence $\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \in \operatorname{Nul} A$.

On the other hand, $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin \operatorname{Nul} A$.

Theorem

Let A be an $m \times n$ matrix.

Then $\mathrm{Nul}\, A$ is a subspace of $\mathbb{R}^n$.

This follows easily from familiar properties of matrix multiplication!

Namely,

- $A0 = 0$,
- $A(x + y) = Ax + Ay$, and
- $A(cx) = c(Ax)$

for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Curious contrast

- Given a matrix A, it is very easy to test if a given vector x belongs to $\mathrm{Nul}\, A$.

  (Just multiply A and x!)

- But how can we find non-zero vectors in $\mathrm{Nul}\, A$ or prove that none exist?