MA334 - Geometry - Problem sheet

(I). Axiom systems

1. Describe the group of symmetries of a non-square rectangle. Calculate it's multiplication table.
2. Calculate the multiplication table for the group of symmetries of an ellipse that is not a circle. How does it compare to the multiplication table of the group of symmetries of a non-square rectangle?
3. Describe the symmetries of a square (There are eight). A group of symmetries $(G, \circ)$ is abelian if $x \circ y = y \circ x$ for any two symmetries $x, y \in G$ (In other words it doesn't matter which symmetry is applied first). Give an example to show that the group of symmetries of a square is not abelian.
4. What are the symmetries of a circular disc? (Note: this group is not finite.) Can you describe the 'multiplication' in this group?
5. A *metric space* (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$ such that the following axioms are satisfied:

   (i) $d(x, y) \geq 0$ for all $x, y \in X$,

   (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

   (iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

   Find a model for this axiom system. That is, describe a set X and function d such that the axioms are satisfied.
6. Calculate $2 + 3$ using the definition of addition using Peano's axioms.
7. Assuming you know the answer to addition problems, calculate 2×3 using the definition of multiplication using Peano's axioms. (i.e., it's ok to write $2 + 2 = 4$ rather than finding $2 + 2$ using Peano's axioms like the previous question)

(II). Hyperbolic geometry

1. Calculate the hyperbolic distance between P and Q and compare it with the Euclidean distance:
   (a) $P = (0, 0),\ Q = (0, 1/2)$;
   (b) $P = (0, 0),\ Q = (0, 3/4)$;
   (c) $P = (0, 1/2),\ Q = (0, 3/4)$;

2. Circles in the Hyperbolic Plane look like Euclidean circles, but the centre is different. Sketch the circle that has the line segment from $(0, 0)$ to $(0, 3/4)$ as a diameter and mark the centre. Now find the hyperbolic centre of this circle. (You have to find the point on the diameter that is half way between the two endpoints.)

3. Suppose that $(0, 2/3)$ is the hyperbolic centre of a circle, and one end point of the diameter is (0,0). Find the other end point of the diameter containing $(0, 0)$, and sketch the circle.

4. With the aid of sketches, verify Euclid's first axiom in the Poincaré disc model for hyperbolic geometry. To help convince yourselves, try using Geogebra to construct any two distinct hyperbolic lines through two distinct points. Use the following steps.
   - Create the disc, i.e., the circle of radius 1 centred at $(0, 0)$.
   - Pick two points P and Q at random inside the disc, draw the line segment between them and find the midpoint M.
   - Draw the line ℓ perpendicular to PQ through M.
   - Pick any point C on ℓ, and draw the circle centred at C through P and Q.
   - Where the circle intersects the unit disk, mark the intersection points and draw the tangents to both circles at those points.
   - Measure the angle between the tangent lines at each intersection point. Observe that they are congruent.
   - Using the cursor, move the point C along the line ℓ, and observe the movement of the tangent lines, the intersection point of the circles, and the angle between the tangent lines. Locate the unique position for C such that these angles are $90°$.

(III). Projective geometry

1. How many different projective points in the real projective plane are represented by the following points in $\mathbb{R}^3$?

$$(1,2,3),\ (2,0,1),\ (-2,-4,-6),\ (-1,2,-3),$$
$$(-4,0,-2),\ (1,0,2),\ (1,0,1/2),\ (3,6,9).$$

2. Consider the following projective lines:

$$x - y + 2z = 0$$
$$x - 2y - 3z = 0$$

Find their point of intersection in $\mathbb{RP}^2$.

3. Let P_1 be the projective point defined by the line through O in $\mathbb{R}^3$ that passes through $(1,1,1)$ and let P_2 be the projective point defined by the line through $(-1,4,0)$. Find the projective line determined by P_1 and P_2.

4. Try to draw a picture of the following:

   (i) the points in $\mathbb{R}^3$ with y-coordinate equal to 1 form a plane that is parallel to the xz-plane and one unit away from it.

   (ii) In this plane, consider the vertical lines that pass through the points $(0,1,0)$ and $(1,1,0)$. These lines are parallel and so they do not meet in this plane.

   Now we are going to think of this picture as a plane projection of the Real Projective Plane $\mathbb{RP}^2$. The two lines correspond to projective lines and we know that any two projective lines must meet at some projective point. Find the projective point where these two projective lines meet.

5. Decompose each of the following projective transformations into a sequence of translations, scalings and inversions:

   (i) $\frac{2}{x+3}$,

   (ii) $\frac{x-1}{x+2}$,

   (iii) $\frac{2x+1}{x-3}$.

6. Calculate the inverse of $f(x) = \frac{2x+1}{x-3}$ in two ways and check you get the same result:

   (i) Let $y = \frac{2x+1}{x-3}$, and write x in terms of y.

(ii) Use the decomposition from part (iii) of the question above, and compose the easily calculated inverse functions of the factors in reverse order.

7. Take $A = 0$, $B = 2$, $C = 4$, $D = 8$ and calculate the cross ratio $[A, B, C, D]$. Find the images $f(A), f(B), f(C), f(D)$ of these points under the projective transformations:

   (i) $f(x) = \frac{2x+1}{x-3}$,

   (ii) $f(x) = \frac{3x-1}{x-2}$.

   Calculate the cross ratio $[f(A), f(B), f(C), f(D)]$ in each case.

8. A car is travelling along a straight road towards a junction. Before the junction, there are warning signs at distances of 2 km and 3 km. In an aerial photograph, the signs are 4 cm and 6 cm from the junction and the car is 1 cm from the junction. How far is the car from the junction on the ground?

9. An aerial photograph shows a train travelling between two stations on a straight track. The stations are 40 km apart. On the film, the stations are 8 cm apart, the train has covered three quarters of the distance between the stations and the tracks appear to meet 16 cm beyond the second station. How far is the train away from the second station?

10. Briefly describe the extended Euclidean plane. Verify that it satisfies the axioms:

    (i) Any two points are contained on a unique line.

    (ii) Any two lines intersect in a unique point.

(IV). Spherical geometry

1. Explain why each of these is zero:
   (a) $\mathbf{A} \times \mathbf{A}$.
   (b) $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})$.
   (c) $\mathbf{A} \times ((\mathbf{B} - \mathbf{C}) \times (\mathbf{C} - \mathbf{B}))$.
2. Let $\mathbf{A} = (1, 0, 2)$, $\mathbf{B} = (1, -2, 3)$, $\mathbf{C} = (-1, 1, 2)$. Calculate:
   (a) $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$.
   (b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})$ and $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$.
3. Let $\mathbf{A} = (1, 0, 2)$, $\mathbf{B} = (1, -2, 3)$. Find the angle between $\mathbf{A}$ and $\mathbf{B}$ using:
   (a) The dot product.
   (b) The cross product.
4. Find the angle between the plane containing the vectors $(1, 0, 2)$ and $(1, -2, 3)$ and the plane containing the vectors $(1, 1, 0)$ and $(1, -1, 1)$.
5. Find the angle between the planes $2x + 3y - z = 0$ and $x - 4y + 2z = 0$.
6. Find the shortest distance between Chicago 41.5°N, 87.45°W and Washington 38.55°N, 77°W. Assume the radius of the Earth is 6377km.
7. Do the previous question again, using the following method: Consider the spherical triangle with vertices at Chicago, Washington and the North Pole. Apply the Spherical Cosine Rule to the angle at the North Pole (this angle is easy to find using the longitudinal angles of the other two points.)

(V). Euclidean geometry

1. Assume you have only a straight edge and compass, and you are given a unit length. Explain how to:
   (a) Construct a line segment with length $\sqrt{2}$.
   (b) Construct a line segment with length $\sqrt{3}$.
   (c) Construct a line segment with length $\sqrt{n}$ for any $n \in \mathbb{N}$.
2. Let ABC be a right angled triangle with BC as hypotenuse and let AD be the perpendicular from A to BC. Prove that $|AD|^2 = |BD||DC|$.
3. Assume the following:
   (i) We know how to find the midpoint of a line segment.
   (ii) If two points X and Y on a circle are diametrically opposite, ten for any third distinct point Z on the circle, the triangle XZY is right angled with right angle at Z.
   (iii) Given a line segment of length x, we can construct a square with sides of length x.

   Given only a straight edge and compass, demonstrate how, if given a rectangle with unequal sides, to construct a square with the same area. Hint: Use Question 2.
4. Find the sum of the interior angles in a convex n-sided polygon (in radians).
5. Show that, in a regular pentagon with sides of unit length, each of the diagonals has length $(1+\sqrt{5})/2$. You may assume that $\cos(\frac{2\pi}{5}) = \frac{-1+\sqrt{5}}{4}$.
6. Prove that $\sqrt{p}$ is irrational for any prime number p.
7. With reference to the Gauss-Wantzel Theorem explain why, given that we are currently aware of only 5 distinct Fermat primes, it is known that a regular n-gon is constructible for only 31 distinct odd natural numbers $n > 2$.